

## Introduction

For many centuries, the triangular numbers have fascinated people and cultures all over the world. While the triangular numbers are an intriguing part of number theory, the interest in this family of numbers is by no means limited to mathematicians. This essay will identify the triangular numbers and briefly investigate their history in religion and culture. Furthermore, this paper will discuss the triangular number's role in education and number theory.

## What are the Triangular Numbers?

The triangular numbers are actually a family of numbers. They are figurate numbers meaning that they can be represented as a regular and discrete geometric pattern ("Figurate"). Over the centuries the triangular numbers have both captivated and puzzled scholars all over the world. The triangular numbers are so named because they represent the number of dots or other objects that make up the shape of an equilateral triangle. For example:

Number ( $n$ )	1	2	3	4
Triangle	*	* * *	* * * * * *	* * * * * * * * * *
Number of *s	1	1+2	1+2+3	1+2+3+4
Triangular Number ( $T$ )	1	3	6	10

Notice that each triangle is made up of dots that increase by the number  $n$  and the sum of all the numbers that preceded it. For example when  $n$  is 3, the triangular number ( $T$ ) is  $1+2+3=6$ . This generalized formula is valid for any value of  $n$  that you chose. As an

illustration, when  $n=6$ , the triangular number  $(T) = 1+2+3+4+5+6 = 21$ . This simple formula

can also be represented by the series  $T_n = \sum_{i=1}^n i$  (Krowne).

## A Brief History of the Triangular Numbers

While it is unknown to history exactly who first discovered the triangular numbers, it is known that Pythagoras, of ancient Greece, was aware of the triangular numbers as early as 496 BCE (Weisstein). Another ancient reference to the triangular numbers is in the bible. The New Testament cites the triangular numbers on two remarkable occasions. For example, “Simon Peter went up, and drew the net to land full of great fishes, an hundred and fifty and three: and for all there were so many, yet was not the net broken” (King James Version, John 21.11). The number 153 is the triangular number for 17. Many Christians believe that the number 153 symbolizes Jesus. In contrast, the number 666 is also mentioned in the New Testament in the Book of Revelation. “Here is wisdom. Let him that hath understanding count the number of the beast: for it is the number of a man; and his number [is] Six hundred threescore [and] six” (Revelation 13.18). It should be noted that theologians believe that the New Testament book of John and the book of Revelation were written by the same author, the Apostle John. Clearly the Apostle John was aware of the triangular numbers. Some Christians believe that there is a divine explanation as to why the numbers 153 and 666 were mentioned so prominently in the bible.

Christian mathematicians have taken these two triangular numbers and developed what they call the “Trinity Function.” The trinity function is the sum of individual numbers within a number raised to the 3rd power (McMurray). Observe the results of the Trinity Function as it is applied to the number 666 five times.

The trinity function of 666 is:

$$\begin{array}{r} 6^3 \\ 216 + \end{array} \quad \begin{array}{r} 6^3 \\ 216 + \end{array} \quad \begin{array}{r} 6^3 \\ 216 \end{array} \quad =648$$

The trinity function of 648 is:

$$\begin{array}{r} 6^3 \\ 216 + \end{array} \quad \begin{array}{r} 4^3 \\ 64 + \end{array} \quad \begin{array}{r} 8^3 \\ 512 \end{array} \quad =792$$

The trinity function of 792 is:

$$\begin{array}{r} 7^3 \\ 343 + \end{array} \quad \begin{array}{r} 9^3 \\ 729 + \end{array} \quad \begin{array}{r} 2^3 \\ 16 \end{array} \quad =1080$$

The trinity function of 1080 is:

$$\begin{array}{r} 1^3 \\ 1 + \end{array} \quad \begin{array}{r} 8^3 \\ 512 \end{array} \quad =513$$

The trinity function of 513 is:

$$\begin{array}{r} 5^3 \\ 125 + \end{array} \quad \begin{array}{r} 1^3 \\ 1 + \end{array} \quad \begin{array}{r} 3^3 \\ 27 \end{array} \quad =153$$

As illustrated above, if you apply so called trinity function 5 times to the number 666, the result is 153. Some Christians believe that this represents that Jesus (153) will ultimately be victorious over Satan (666). Interestingly, when the trinity function is applied to 153 the results are 153. Furthermore, when the trinity function is applied to any number that is divisible by 3 the eventual outcome will be 153. Also, for any number divisible by 3 and less than  $10^8$ , it will take 14 or fewer cycles of the trinity function to reach 153 (Wells, 125). Clearly the Apostle John was aware of the triangular numbers, and he may have even been aware of the trinity function. As a result, since most historians believe that the New Testament of the bible was written between 60 and 100 CE, these biblical references are among the earliest known

recordings of the triangular numbers. Jumping ahead in time, there are many other references to the triangular numbers in history.

The triangular numbers are an integral part of an array of numbers that we today call Pascal's Triangle. Each term of the array is the sum of the two numbers on the flanks above the number (Pappas, 134):

			1											
			1		1									
		1		2		1								
		1		<b>3</b>		<b>3</b>		1						
		1		4		<b>6</b>		4		1				
		1		5		<b>10</b>		<b>10</b>		5		1		
		1		6		<b>15</b>		20		<b>15</b>		6		1

Notice that the triangular numbers, in bold, are located in the third diagonal row from both the right and left sides of the triangle. In addition to the triangular numbers, this array contains other families of numbers such as the natural numbers, the tetrahedral numbers, the pentatonic numbers, and many others.

It follows that those who studied Pascal's Triangle also studied the triangular numbers. Pascal's Triangle is named for Blaise Pascal (1623-1662) because he is well known for devoting extensive study and research into the triangle. He was especially interested in binomial expansion and probability. Today most people attribute the triangle to Pascal. However, it is well known that the array we currently attribute to Blaise Pascal was actually studied long before his birth.

Pascal's Triangle was studied by the famous Persian mathematician Omar Khayyam and by Chinese mathematicians around the 12<sup>th</sup> century CE. Is it just coincidence that the Chinese and the Persians studied the array at the same time? In the 12<sup>th</sup> century there was not extensive communication between China and Persia, but a silk route did exist. As a result, it is possible that the array was shared between the two regions (Boyer, 242). Omar Khayyam died in 1123 CE—this is hundreds of years before the birth of Pascal. (Conway and Guy 68).

Hence, Pascal's Triangle, and as a result the triangular numbers, have been studied by many cultures and on several continents for centuries. Perhaps if it were not for Euro-centrism, Pascal's Triangle would be known as Omar Khayyam's Triangle. Still, Monsieur Pascal certainly deserves much credit for his work with the array of numbers that today bears his name. Below is a print from Blaise Pascal's Treatise on the Arithmetic Triangle from 1654.

Note that the triangular numbers are listed in the third horizontal row from the top.

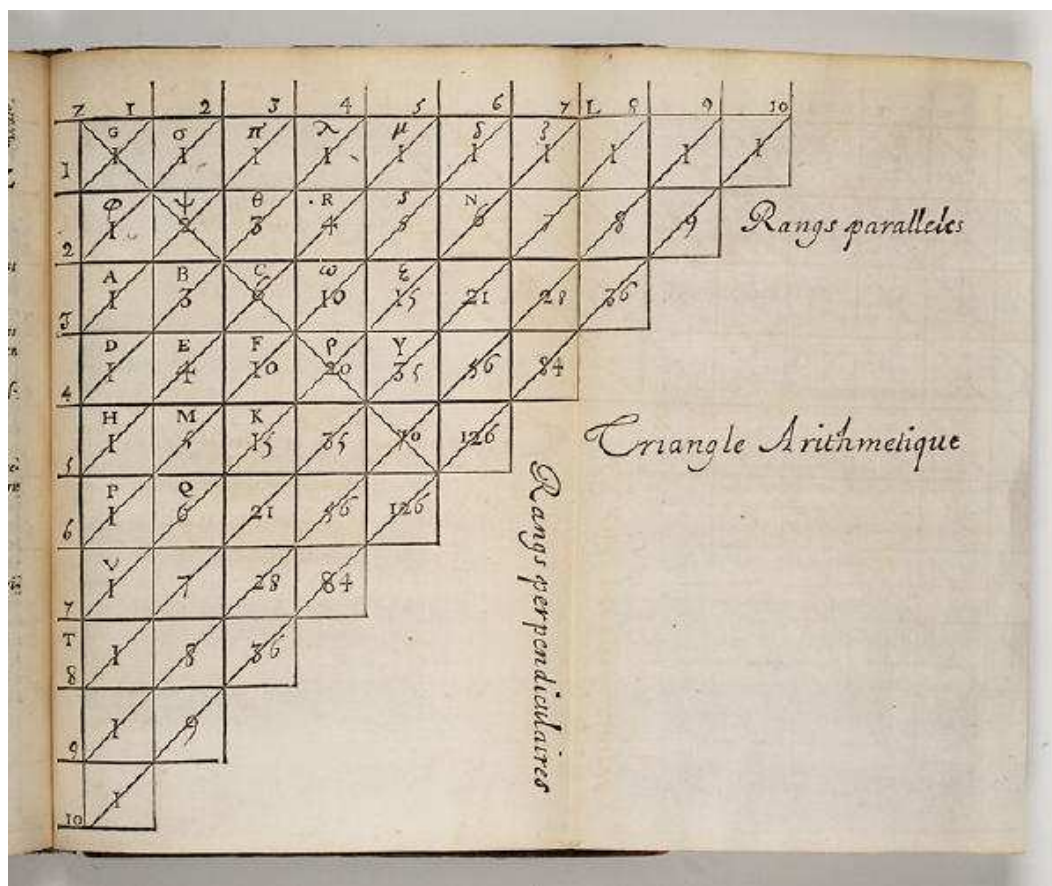


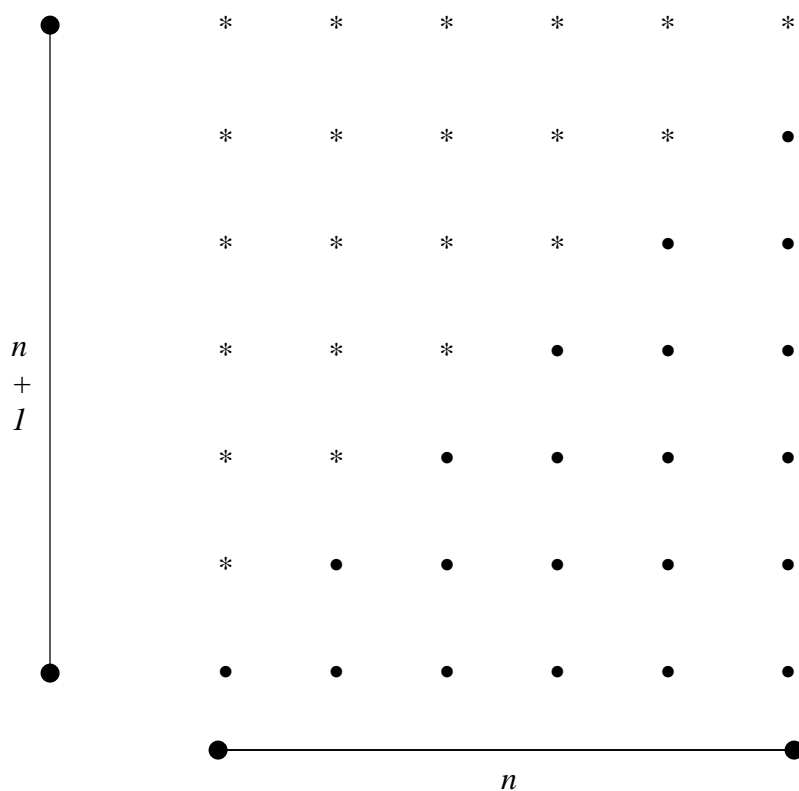
Image from the Cambridge University Library <http://www.lib.cam.ac.uk/cgi-bin/PascalTriangle/browse?4>

Jumping ahead to the present day, the triangular numbers have become an integral part of education.

## Triangular Numbers in Education

The triangular numbers are often used by middle school educators as a way to introduce their students to number theory. As often is the case with middle school students, one of the most effective ways to teach is often by way of a group activity. As an illustration, let's look at a method that is used by some teachers to introduce the triangular numbers to students. A teacher provides his students with pennies, bingo chips, or any other similar type objects and has them separate into work groups. The teacher asks his students to make triangles of different

sizes. The students are told to carefully note the number of pennies it takes to make each triangle. This activity leads to a discussion of the triangular numbers. The students learn that the total number of pennies it takes to make each triangle is the triangular number ( $T$ ). The teacher challenges the groups to find numerical patterns in their triangles. For example, the teacher may ask “how many pennies do you have to add to the bottom row to make the triangle larger?” Follow up questions could be “how many total pennies would you need to make a triangle with a base (bottom row) of six pennies?” The teacher explains that the number of pennies in the bottom row is the  $n$  number and the total number of pennies in the triangle is  $T$  (Warner). This could lead to a discussion of how a formula for calculating the triangular numbers is obtained. Let’s now look at an intuitive method for obtaining a formula for the triangular numbers.



Using geometric patterns it is easy to obtain a formula for producing the triangular number ( $T$ ) for any given value of  $n$ . Simply take two triangles of the same size and place them together as illustrated above. In the above figure, each individual triangle is made up of the symbols \* or •. Notice that this creates a rectangle with the length ( $l$ ) represented as  $(n+1)$  and a width ( $w$ ) of  $n$ . Recall that the formula for the area of a rectangle is  $a=lw$ . So, the area of the rectangle in the above figure is  $n(n+1)$ . However, we are interested in the triangular numbers—which is exactly half of our “rectangle.” So we simply take half the area of the rectangle. This gives us the following formula to obtain the triangular numbers:

$$T(n) = \frac{n(n+1)}{2}. \text{ (Conway and Guy page 34).}$$

## Triangular Numbers in Number Theory

One of the many fascinating aspects of the triangular numbers is that the sum of any two consecutive triangular numbers is a perfect square. Notice that the sums of first five pairs of triangular numbers are perfect squares:

a)  $1+3=4$

b)  $6+10=16$

c)  $10+15=25$

d)  $21+28=49$

e)  $36+45=81$

This pattern of the sums of two consecutive triangular numbers adding up to a perfect square continue as the numbers get larger and larger. Notice the following larger examples:

- a)  $55+66=121$        $\sqrt{121} = 11$
- b)  $4950+5050=10,000$        $\sqrt{10,000} = 100$
- c)  $19,900+20,100=40,000$        $\sqrt{40,000} = 200$
- d)  $31,626+31,878=63,504$        $\sqrt{63,504} = 252$

In fact, this pattern continues to infinity. This pattern can be summarized by the following formula: Let  $T_n$  represent the  $n$ th triangular number then  $T_{n-1} + T_n = n^2$  (Numsen).

Another interesting aspect of the triangular numbers is that they are in consecutive pairs of alternating odd and even integers. The following table of triangular numbers illustrates this fact:

1	3	6	10	15	21
28	36	45	55	66	78
91	105	120	136	153	171
190	210	231	253	276	300
325	351	378	406	435	465

This writer found that the pattern is a result of a simple property of arithmetic. Recall that the sum of an odd and an even integer is always odd; the sum of an odd integer and an odd integer is always even; and, the sum of an even number and an even number is always even. Remember that the triangular numbers are formed by adding consecutive positive whole numbers i.e.  $T_3 = 1 + 2 + 3 = 6$ . So keeping in mind the property that determines whether or not a number is odd or even we have:

$$T_1 = 1 + 0 = 1$$

$$(\text{odd} + \text{even}) = \text{odd}$$

$$T_2 = 1 + 2 = 3$$

$$(\text{odd} + \text{even}) = \text{odd}$$

$$T_3 = (1 + 2) + 3 = 6$$

$$(\text{odd}) + \text{odd} = \text{even}$$

$$T_4 = (1 + 2) + (3 + 4) = 10$$

$$(\text{odd}) + (\text{odd}) = \text{even}$$

$$T_5 = (1 + 2) + (3 + 4) + 5 = 15$$

$$(\text{odd}) + (\text{odd}) = \text{even} + (\text{odd}) = \text{odd}$$

$$T_6 = (1 + 2) + (3 + 4) + (5 + 6) = 21$$

$$(\text{odd}) + (\text{odd}) = \text{even} + (\text{odd}) = \text{odd}$$

$$T_7 = (1 + 2) + (3 + 4) + (5 + 6) + 7 = 28$$

$$(\text{odd}) + (\text{odd}) = \text{even} + (\text{odd}) = \text{odd} = \text{even}$$

$$T_8 = (1 + 2) + (3 + 4) + (5 + 6) + (7 + 8) = 36$$

$$(\text{odd}) + (\text{odd}) = \text{even} + (\text{odd}) = \text{odd} + (\text{odd}) = \text{even}$$

Notice how the pattern develops. For example when  $n=4$  we have

$T_4 = (1 + 2) + (3 + 4) + 5 = 15$ . Notice how the addends are grouped together in pairs. The order of operations requires that we add from left to right. So,  $T_4 = (\text{odd}) + (\text{odd}) = (\text{even}) + (\text{odd}) = \text{odd}$ . When  $T_5 = (1 + 2) + (3 + 4) + (5 + 6) = 21$ . Again we have  $T_5 = (\text{odd}) + (\text{odd}) = (\text{even}) + (\text{odd}) = \text{odd}$ . This leads to the following formula that generalizes the parity of the triangular numbers:

Let  $T_n$  and  $T_{n+1}$  = a consecutive pair of triangular numbers. Then both  $T_n$  and  $T_{n+1}$  are odd numbers where  $n=1$  and every other odd positive integer beginning with the number 1.

Conversely, both  $T_n$  and  $T_{n+1}$  are even numbers where  $n=3$  and every other odd positive integer beginning with the number 3.

The fact that the triangular numbers are such a great part of number theory is based in part upon the fact that all known perfect numbers are also triangular numbers (Gupta). We will discuss the validity of this assertion in the following paragraphs. Recall that a perfect number is a number that is equal to the sum of all its positive divisors except for the number itself (Papick, 79). Remember also Euclid's method to obtain perfect numbers is to sum consecutive powers of 2 beginning at  $2^0$ ; then, check to see if this sum is a prime number. If the sum is prime, multiply it by the highest power of 2 and the resulting product will be a perfect number. For example,  $2^0 + 2^1 = 3$ ; since the sum (3) is prime, multiply it by the highest power of 2 that was within the sum. In this case,  $2^1 * 3 = 6$ . It is well known that 6 is the first perfect number. In summary, Euclid's method for obtaining perfect numbers requires two factors: a power of 2 and a prime factor. We will now apply Euclid's method to the triangular numbers to confirm that the first five known perfect numbers are also triangular numbers.

The first 3 perfect numbers have been known since antiquity; they are 6, 28, and 492. In more recent times, many new perfect numbers have been discovered. Today it is widely known that the next two perfect numbers are: 8128 and 33,550,336. Using the general formula for the triangular numbers  $T_n = \frac{n^2 + n}{2}$  to obtain the first few triangular numbers we observe immediately that  $T_3 = 6$ . Please note from general formula above

that the prime factor using Euclid’s method to obtain the perfect number 6 is the same as the  $n$  number used in the general formula, 3. We will now observe if the prime factor, from Euclid’s method, also returns a perfect number when plugged into the general triangular number formula. We will also confirm the first five known perfect numbers using Euclid’s method.

Sum	Prime Factor	Sum* Highest Power of 2	Perfect number	$T(\text{prime factor}) = \frac{n^2 + n}{2}$
$2^0 + 2^1 = 3$	3	$3 * 2^1$	6	6
$2^0 + 2^1 + 2^2 = 7$	7	$7 * 2^2$	28	28
$2^0 + 2^1 + 2^2 + 2^3 + 2^4 = 31$	31	$31 * 2^4$	496	496
$2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 = 127$	127	$127 * 2^6$	8128	8128
$2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 \dots\dots + 2^{12} = 8191$	8191	$8191 * 2^{12}$	33,550,336	33,550,336

From the table above, observe that  $T_{\text{PrimeFactor}}$  returns the same number as that returned by Euclid’s method for obtaining perfect numbers. Therefore, at least the first five perfect numbers are also triangular numbers. However, after observing the above table, some observations come to mind.

One may well notice that the prime factors from the above table are actually Mersenne Prime numbers. Recall that a Mersenne number is a number that is one less than a power of two. A Mersenne number is often expressed in the form of  $2^m - 1$ . One may also notice that the highest power of two from Euclid’s method is actually a power one less than the Mersenne number. (This can be noted as  $2^{m-1}$ .) This leads to a very important characterization that was made by Dr. Leonard Euler during the nineteenth century. It states that an even number  $n$  is perfect if and only if  $n = 2^{m-1} (2^m - 1)$ , where  $2^m - 1$  is

prime (Papick, 83). Please observe the table below for an example of Dr.Euler's characterization of the first five prime numbers:

<b>Prime Factor Mersenne Number</b>	$2^{m-1}$	$(2^m - 1) * 2^{m-1}$	<b>Perfect Number</b>
$3=2^2 - 1$	2	$3*2$	6
$7=2^3 - 1$	4	$7*4$	28
$31=2^5 - 1$	16	$31*16$	496
$127=2^7 - 1$	64	$127*64$	8128
$8191=2^{13} - 1$	4096	$8191*4096$	33,550,336

We have shown that the first five perfect numbers are indeed triangular numbers.

While these observations do not conclusively prove that all perfect numbers are also triangular numbers, there does appear to be substantial evidence that this may well be the case.

The triangular numbers have fascinated people all over the world for many centuries. Their unique properties make them an integral part of education, history, religion, and number theory. There is little doubt that people will continue to study the triangular numbers for many more centuries.

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