Theorem 1. *Let*

$$\mathbf{T} = \left(\begin{array}{c|c} f & \mathbf{c}^T \\ \hline \mathbf{b} & \mathbf{A} \end{array}\right)$$

be a tableau, let $\mathbf{B} = (\mathbf{A}_{j_1} \cdots \mathbf{A}_{j_m})$ *with columns* $\mathbf{A}_{j_1}, \ldots, \mathbf{A}_{j_m}$ *linearly independent and let* $\mathbf{X}$ *be a non-singular matrix with first column* $(1, 0, \ldots, 0)^T$*. If*

$$\mathbf{X}\mathbf{T} = \widetilde{\mathbf{T}} = \left(\begin{array}{c|c} \widetilde{f} & \widetilde{\mathbf{c}}^T \\ \hline \widetilde{\mathbf{b}} & \widetilde{\mathbf{A}} \end{array}\right)$$

is such that

$$\widetilde{\mathbf{T}}_B = [\widetilde{\mathbf{T}}_{j_1} \cdots \widetilde{\mathbf{T}}_{j_m}] = \left(\begin{array}{c} \mathbf{0}^T \\ \hline \mathbf{I} \end{array}\right),$$

then

$$\widetilde{\mathbf{T}} = \left(\begin{array}{c|c} f - \mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{b} & \mathbf{c}^T - \mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{A} \\ \hline \mathbf{B}^{-1}\mathbf{b} & \mathbf{B}^{-1}\mathbf{A} \end{array}\right).$$

and

$$\mathbf{X} = \left(\begin{array}{c|c} 1 & -\mathbf{c}_B^T \mathbf{B}^{-1} \\ \hline \mathbf{0} & \mathbf{B}^{-1} \end{array}\right)$$

where $\mathbf{c}_B^T = (c_{j_1}, \ldots, c_{j_m})^T$.

Remark.

An important corollary is that for each $1 \leq j \leq n$,

$$(*) \qquad \widetilde{c}_j = c_j - \mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{A}_j$$

and

$$(*) \qquad \widetilde{\mathbf{A}}_j = \mathbf{B}^{-1}\mathbf{A}_j.$$

In the simplex procedure there are two row operations: either we multiple some row $i \in \{1, \ldots, n\}$ by α, or we add some multiple of a row $i \in \{1, \ldots, n\}$ to a row $j \in \{0, \ldots, n\}$. The first operation corresponds to pre-multiplication of $\mathbf{T}$ by $\mathbf{I}(i, \alpha)$ which is obtained from the identity matrix $\mathbf{I}$ by replacing 1 in the entry (i, i) with α, and the second operation corresponds to pre-multiplication of $\mathbf{T}$ by $\mathbf{I}(i, j, \alpha)$ which is obtained from the identity matrix $\mathbf{I}$ by replacing 0 in the entry (i, j) with α. This means that $\widetilde{\mathbf{T}}$ is obtained from $\mathbf{T}$ by a sequence of pre-multiplications by such matrices. Note that in all cases the matrix we pre-multiply by has the first column $(1, 0, 0, \ldots, 0)^T$, i.e. we never add a multiple of row 0 to row i where $i \in \{1, \ldots, n\}$ and we never multiply row 0 by a constant. Therefore, the product of these matrices has first column $(1, 0, 0, \ldots, 0)^T$ since the product of two matrices with the zero column $(1, 0, 0, \ldots, 0)^T$ also has the zero column $(1, 0, 0, \ldots, 0)^T$.

Proof.

Claim. *The inverse of a matrix* $\mathbf{X}$ *with the first column* $(1,0,0,\ldots,0)^T$ *also has the first column* $(1,0,0,\ldots,0)^T$.

Proof. Let $\mathbf{A}$ be a matrix with m columns. If $\mathbf{X}$ has first column $(1,0,0,\ldots,0)^T$, then the first column of $\mathbf{AX}$ is simply the first column of $\mathbf{A}$. Since $\mathbf{X}^{-1}\mathbf{X} = \mathbf{I}$, the first column of $\mathbf{X}^{-1}$ is the first column of $\mathbf{I}$: $(1,0,0,\ldots,0)^T$. $\square$

Let $\mathbf{X} = \left(\begin{array}{c|c} 1 & \mathbf{u}^T \\ \hline \mathbf{0} & \mathbf{U} \end{array}\right)$ and by the claim we can let $\mathbf{X}^{-1} = \left(\begin{array}{c|c} 1 & \mathbf{w}^T \\ \hline \mathbf{0} & \mathbf{W} \end{array}\right)$. Note that $\mathbf{X}^{-1}\mathbf{X} = \mathbf{I}$ implies that $0 = \mathbf{u}^T + \mathbf{w}^T\mathbf{U}$ and $\mathbf{WU} = \mathbf{I}$. So $\mathbf{U} = \mathbf{W}^{-1}$ and $\mathbf{u}^T = -\mathbf{w}^T\mathbf{U}$.

We have that $\mathbf{T} = \mathbf{X}^{-1}\widetilde{\mathbf{T}}$. Therefore,

$$\left(\begin{array}{c} \mathbf{c}_B^T \\ \hline \mathbf{B} \end{array}\right) = \mathbf{T}_B = \mathbf{X}^{-1}\widetilde{\mathbf{T}}_B = \left(\begin{array}{c|c} 1 & \mathbf{w}^T \\ \hline \mathbf{0} & \mathbf{W} \end{array}\right) \left(\begin{array}{c} \mathbf{0}^T \\ \hline \mathbf{I} \end{array}\right) = \left(\begin{array}{c} \mathbf{w}^T \\ \hline \mathbf{W} \end{array}\right)$$

So $\mathbf{w} = \mathbf{c}_B$ and $\mathbf{W} = \mathbf{B}$. This implies $\mathbf{U} = \mathbf{B}^{-1}$ and $\mathbf{u}^T = -\mathbf{c}_B^T\mathbf{B}^{-1}$. Now that we know $\mathbf{X}$, we get the formula for $\widetilde{\mathbf{T}}$, as well. $\square$