Two-phase simplex method

Suppose we are given an LP in standard form

$$\text{Minimize } z = \mathbf{c}^T \mathbf{x}$$

subject to
$$(*) \qquad \begin{cases} \mathbf{Ax} = \mathbf{b}, \\ \mathbf{x} \geq \mathbf{0}. \end{cases}$$

As before, we assume that the rank of the system is m, since otherwise we simply delete some dependent equations. We also may assume that $b_i \geq 0$ for each i, since otherwise we can multiply the corresponding equation by -1

First, we digress about *basic feasible solutions* of (*). A *basis of* (*) is a set $\mathbf{B}$ of m linearly independent columns of $\mathbf{A}$. We often view $\mathbf{B}$ as a square non-degenerate submatrix of $\mathbf{A}$. We also associate with $\mathbf{B}$ the variables corresponding to the columns of $\mathbf{B}$. A *basic feasible solution of (*) corresponding to a basis* $\mathbf{B}$ is a solution of (*) such that
(i) every variable not corresponding to a column in $\mathbf{B}$ equals 0;
(ii) the value of every variable corresponding to a column in $\mathbf{B}$ is nonnegative.

Note that under condition (i), the system $\mathbf{Ax} = \mathbf{b}$ becomes $\mathbf{B}\tilde{\mathbf{x}} = \mathbf{b}$, where $\tilde{\mathbf{x}}$ is the truncation of $\mathbf{x}$ corresponding to $\mathbf{B}$. Since $\mathbf{B}$ is non-degenerate, there is only one solution of such a system. Observe that this solution is not a convex combination of any other feasible solutions of (*). That is, it is a "corner" of the set of feasible solutions of (*).

We already know that once we have a b.f.s. for any LP, we can solve it in a finite time. But how do we find out whether our system has feasible solutions at all and if it has, how do we find a b.f.s.?

To do this, we solve an auxiliary LP for which we know from the beginning that it has a b.f.s. The new system of equations is obtained from (*) by adding to each Equation i a summand y_i and by demanding that $y_i \geq 0$ for each i. The new objective function to minimize is $\xi = y_1 + \ldots + y_m$. In other words, we have the following system:

$$\text{Minimize } \xi = y_1 + \ldots + y_m$$

subject to
$$(**) \qquad \begin{cases} (\mathbf{I} \mid \mathbf{A}) \left(\begin{smallmatrix} \mathbf{y} \\ \mathbf{x} \end{smallmatrix}\right) = \mathbf{b}, \\ \mathbf{y} \geq \mathbf{0}, \\ \mathbf{x} \geq \mathbf{0}. \end{cases}$$

The tableau for the new LP will be

		y_1	y_2	$\ldots$	y_m	x_1	$\ldots$	x_n
$y_0 = -\xi$	$a_{0,0}$	1	1	$\ldots$	1	0	$\ldots$	0
y_1	$a'_{1,0}$	1	0	$\ldots$	0	$a'_{1,1}$	$\ldots$	$a'_{1,n}$
y_2	$a'_{2,0}$	0	1	$\ldots$	0	$a'_{2,1}$	$\ldots$	$a'_{2,n}$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
y_m	$a'_{m,0}$	0	0	$\ldots$	1	$a'_{m,1}$	$\ldots$	$a'_{m,n}$

System (**) is of course not equivalent to (*). A good feature of (**) is that it has the basic feasible solution: $x_j = 0$ for $j = 1, \ldots, n$ and $y_i = a'_{i,0}$ for $i = 1, \ldots, m$. Furthermore, since we minimize the sum $y_1 + \ldots + y_m$ of nonnegative variables, the objective function is bounded from below, and hence after a finite number of steps we will find the optimal value of ξ. Furthermore, if (*) has any feasible solution, then (**) has a solution with $\xi = 0$. Thus, if the optimal solution of (**) is not 0, we conclude that (*) has no feasible solutions at all.

Suppose that $\xi = 0$. Then we have some feasible solution of (*). But we want a basic feasible solution. If none of y_i is in the basis, then we do have a b.f.s. for (*). Suppose that, say y_1, is the basic variable corresponding to Row 1. Since $\xi = 0$, we have $a'_{1,0} = 0$. Since the rank of A was m, the equation contains a non-zero coefficient at some x_j. Since, the coefficient is non-zero, x_j is not in the basis. So, we pivot on $a'_{1,j}$. It may spoil lexicography, but none of $a'_{i,0}$ has changed (since $a'_{1,0} = 0$) and hence stays non-negative. We can do this consecutively for every y_i that still is in the basis and finally get a basis consisting only from x_j. Recall that no $a'_{i,0}$ changes during this procedure. Then we simply delete all columns corresponding to $y_1, \ldots, y_m$ and move the columns corresponding to basic variables in front of other columns. This way, we will get all rows (apart from Row 0) lexicographically positive, and can start our basic algorithm (with replacing back the objective function).

Altogether, we have proved the following theorem.

Theorem 1 *For every linear program P exactly one of the following alternatives holds:*
(a) there are no feasible solutions,
(b) there are solutions with arbitrarily small $z = \mathbf{c}^T \mathbf{x}$,
(c) the minimum of z iz finite and there exists an optimum **basic feasible solution**.