Matrix games

A vector $\mathbf{y} \in R^m$ is *stochastic* if $y_i \geq 0$ for every $i \in \{1, \ldots, m\}$ and $\sum_{i=1}^m y_i = 1$. Throughout, assume $\mathbf{x}, \mathbf{y}$ are stochastic vectors in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $A = \{a_{ij}\}$ be an $m \times n$ payoff matrix for a game with zero sum. If the first player chooses his/her strategy i with probability y_i for every $i = 1, \ldots, m$, and the second player chooses his/her strategy j with probability x_j for all $j = 1, \ldots, n$ then the expectation of the profit of the first player will be

$$F(A, \mathbf{y}, \mathbf{x}) = \sum_{i=1}^m \sum_{j=1}^n a_{ij} y_i x_j = \mathbf{y}^T A \mathbf{x}.$$

Thus the first player can provide the expected profit $v_1(A) = \max_{\mathbf{y}} \min_{\mathbf{x}} F(A, \mathbf{y}, \mathbf{x})$ and the second player's expected loss can be made at most
$v_2(A) = \min_{\mathbf{x}} \max_{\mathbf{y}} F(A, \mathbf{y}, \mathbf{x})$. It is not hard to see that $v_1(A) \leq v_2(A)$ for every payoff matrix A.

We need the following lemma.

Lemma. *For any payoff matrix A and stochastic $\mathbf{y} \in \mathbb{R}^m$, $\min_{\mathbf{x}} \mathbf{y}^T A \mathbf{x} = \min_j \sum_{i=1}^m y_i a_{i,j}$. And for any stochastic $\mathbf{x} \in \mathbb{R}^n$, $\max_{\mathbf{y}} \mathbf{y}^T A \mathbf{x} = \max_i \sum_{j=1}^n a_{i,j} x_j$,*

Proof. Let $t = \min_j \sum_{i=1}^m y_i a_{i,j}$. We have that

$$\mathbf{y}^T A \mathbf{x} = \sum_{j=1}^n \sum_{i=1}^m y_i a_{i,j} x_j = \sum_{j=1}^n x_j \sum_{i=1}^m y_i a_{i,j} \geq \sum_{j=1}^n x_j t = t,$$

so $\min_{\mathbf{x}} \mathbf{y}^T A \mathbf{x} \geq t$. Furthermore, for any $j \in \{1, \ldots, n\}$,

$$\min_{\mathbf{x}} \mathbf{y}^T A \mathbf{x} \leq \mathbf{y}^T A \mathbf{e_j} = \sum_{i=1}^m y_i a_{i,j},$$

where $\mathbf{e_j} \in \mathbb{R}^n$ is the jth standard basis vector. Hence, $\min_{\mathbf{x}} \mathbf{y}^T A \mathbf{x} \leq \min_j \mathbf{y}^T A \mathbf{e_j} = t$ and the conclusion follows.

The proof of the second sentence is similar. $\square$

Theorem. *For every payoff matrix A, $v_1(A) = v_2(A)$.*

PROOF. Consider the following LP1:

$$\begin{array}{llllllll}
 & & \text{Find} & \max v_1 & & & & \\
\text{such that} & v_1 & -a_{11}y_1 & -a_{21}y_2 & \ldots & -a_{m1}y_m & \leq & 0 \\
 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
 & v_1 & -a_{1n}y_1 & -a_{2n}y_2 & \ldots & -a_{mn}y_m & \leq & 0 \\
 & & y_1 & +y_2 & \ldots & +y_m & = & 1 \\
 & & & & & v_1 & & \text{unconstrained} \\
 & & & & & y_i & \geq & 0 \quad \forall i
\end{array}$$

Using the lemma, one can check that the maximum possible v_1 in this LP is exactly $v_1(A)$.

Similarly, $v_2(A)$ is the solution of the the following LP2:

$$
\begin{array}{lcccccccl}
 & & & \text{Find} & \min v_2 & & & & \\
 & x_1 & x_2 & \dots & x_n & & = & 1 & \\
 & -a_{11}x_1 & -a_{12}x_2 & \dots & -a_{1n}x_n & +v_2 & \geq & 0 & \\
\text{such that} & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\
 & -a_{m1}x_1 & -a_{m2}x_2 & \dots & -a_{mn}x_n & +v_2 & \geq & 0 & \\
 & & & & & x_j & \geq & 0 & \forall j \\
 & & & & & v_2 & & \text{unconstrained} &
\end{array}
$$

Both these problems have feasible solutions (any pure strategies would do). Moreover, they are DUAL. This proves the theorem.