An example of the primal–dual simplex method

Suppose we are given the problem **P**:

$$\text{Minimize } z = x_1 + 3x_2 + 3x_3 + x_4$$

$$\text{subject to} \quad \begin{cases} 3x_1 & +4x_2 & -3x_3 & +x_4 & = & 2, \\ 3x_1 & -2x_2 & +6x_3 & -x_4 & = & 1, \\ 6x_1 & +4x_2 & & +x_4 & = & 4 \\ x_1, & x_2, & x_3, & x_4 & \geq & 0. \end{cases} \tag{1}$$

The dual to **P** is of course the following **D**

$$\text{Maximize } w = 2\pi_1 + \pi_2 + 4\pi_3$$

$$\text{subject to} \quad \begin{cases} 3\pi_1 & +3\pi_2 & +6\pi_3 & \leq & 1, \\ 4\pi_1 & -2\pi_2 & +4\pi_3 & \leq & 3, \\ -3\pi_1 & +6\pi_2 & & \leq & 3, \\ \pi_1 & -\pi_2 & +\pi_3 & \leq & 1. \end{cases} \tag{2}$$

Somebody tells us that probably vector $\pi = (1/3,\, 0,\, 0)^T$ is an optimal vector in **D**. Note that the value of w with this π is $2/3$. We start checking this version using complementary slackness. First, we plug this vector in **D** and see that it is a feasible vector and only the first inequality is binding. Hence our first set J is $\{1\}$. In particular, if π is an optimal vector in **D**, then in the corresponding optimal vector $\mathbf{x}$ of **P** only coordinate x_1 can be non-zero. We try to find it by solving the following *restricted primal problem* **RP1**:

$$\text{Minimize } \xi = x_1^r + x_2^r + x_3^r$$

$$\text{subject to} \quad \begin{cases} 3x_1 & +x_1^r & & & = & 2, \\ 3x_1 & & +x_2^r & & = & 1, \\ 6x_1 & & & +x_3^r & = & 4, \\ x_1, & x_1^r, & x_2^r, & x_3^r & \geq & 0. \end{cases} \tag{3}$$

Normally, we would use the revised simplex to solve it. But here we will write down all the tableaus. So, the initial tableau is

		x_1	x_1^r	x_2^r	x_3^r
$y_0 = -\xi$	0	0	1	1	1
x_1^r	2	3	1	0	0
x_2^r	1	3	0	1	0
x_3^r	4	6	0	0	1

Excluding x_1^r, x_2^r, and x_3^r from Row 0, we have

		x_1	x_1^r	x_2^r	x_3^r
$y_0 = -\xi$	-7	-12	0	0	0
x_1^r	2	3	1	0	0
x_2^r	1	3	0	1	0
x_3^r	4	6	0	0	1

We pivot on $a_{2,1}$ and get

		x_1	x_1^r	x_2^r	x_3^r
$y_0 = -\xi$	-3	0	0	4	0
x_1^r	1	0	1	-1	0
x_1	1/3	1	0	1/3	0
x_3^r	2	0	0	-2	1

This is the final tableau which proves that our $\pi = (1/3,\ 0,\ 0)^T$ is NOT optimal. But this is not only a negative outcome, since we now know how to improve the π. Our new $\widetilde{\pi}$ will have the form

$$\widetilde{\pi} = \pi + \theta\pi^r, \tag{4}$$

where θ is a positive factor that we will find below and π^r is an optimal vector in the dual **DRP1** to **RP1** which (by definition) is as follows:

$$\text{Maximize } w^r = 2\pi_1^r + \pi_2^r + 4\pi_3^r$$

subject to

$$\begin{cases} 3\pi_1^r + 3\pi_2^r + 6\pi_3^r \le 0, \\ \pi_1^r \le 1, \\ \pi_2^r \le 1, \\ \pi_3^r \le 1. \end{cases}$$

We can find π^r, from the last tableau for **RP1**, where the vector $(0,4,0)$ in Row 0 is (by an algebraic theorem) in fact $(1,1,1) - (\pi_1^r, \pi_2^r, \pi_3^r)$. Hence $(\pi_1^r, \pi_2^r, \pi_3^r) = (1,1,1) - (0,4,0) = (1,-3,1)$. Now we choose the maximum θ such that the vector $\widetilde{\pi}^T = (1/3,0,0) + \theta(1,-3,1)$ is feasible in **D**. Plugging this $\widetilde{\pi}$ into the first inequality of **D** we get the inequality

$$\widetilde{\pi}^T \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix} = (1/3,0,0)\begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix} + \theta(1,-3,1)\begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix} = 1 + \theta \cdot 0 = 1 \le 1,$$

which holds for every θ. Similarly, plugging $\widetilde{\pi}$ into the second inequality of **D** we get the inequality $4/3 + \theta 14 \le 3$ which holds for $\theta \le 5/42$. Plugging $\widetilde{\pi}$ into the third inequality of **D** we get $-1 + \theta(-21) \le 3$ which holds for every positive θ. Finally, plugging $\widetilde{\pi}$ into the fourth inequality of **D** we get $1/3 + \theta \cdot 5 \le 1$ which holds for $\theta \le 2/15$. Thus we choose $\theta = 5/42$ and hence our new π (we omit the tilde) is $(1/3,0,0)^T + \frac{5}{42}(1,-3,1)^T = (\frac{19}{42}, \frac{-5}{14}, \frac{5}{42})^T$. Note that now $w = 2\frac{19}{42} - \frac{5}{14} + 4\frac{5}{42} = \frac{43}{42}$.

So, we start our cycle again. We hope that the new π is optimal. Plugging it in **D** we see that now $J = \{1,2\}$. Thus, our new restricted primal **RP2** is

$$\text{Minimize } \xi = x_1^r + x_2^r + x_3^r$$

subject to

$$\begin{cases} 3x_1 + 4x_2 + x_1^r = 2, \\ 3x_1 - 2x_2 + x_2^r = 1, \\ 6x_1 + 4x_2 + x_3^r = 4, \\ x_1,\ x_2,\ x_1^r,\ x_2^r,\ x_3^r \ge 0. \end{cases} \tag{5}$$

But we do not start from scratch. We use the last tableau of the previous iteration adding there the values of the x_2-column obtained from knowing B^{-1}:

		x_1	x_2	x_1^r	x_2^r	x_3^r
$y_0 = -\xi$	-3	0	-14	0	4	0
x_1^r	1	0	6	1	-1	0
x_1	1/3	1	$-2/3$	0	1/3	0
x_3^r	2	0	8	0	-2	1

Here, the second column was obtained using the formulas $\widetilde{c}_2 = c_2 - (1,\ -3,\ 1)\begin{pmatrix} 4 \\ -2 \\ 4 \end{pmatrix} = 0 - 14 = -14$, and $\widetilde{A}_2 = B^{-1}A_2 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1/3 & 0 \\ 0 & -2 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ -2/3 \\ 8 \end{pmatrix}$. Note that B^{-1} is in the last three rows and columns of the previous tableau.

We pivot on $a_{1,2}$ and get

		x_1	x_2	x_1^r	x_2^r	x_3^r
$y_0 = -\xi$	$-2/3$	0	0	7/3	5/3	0
x_2	1/6	0	1	1/6	$-1/6$	0
x_1	4/9	1	0	1/9	2/9	0
x_3^r	2/3	0	0	$-4/3$	$-2/3$	1

This is the final tableau which proves that our new π again is not optimal. So, we again correct it using (4). Recall that our restricted dual **DRP2** is

$$\text{Maximize } w^r = 2\pi_1^r + \pi_2^r + 4\pi_3^r$$

$$\text{subject to} \quad \begin{cases} 3\pi_1^r & +3\pi_2^r & +6\pi_3^r & \leq & 0, \\ 4\pi_1^r & -2\pi_2^r & +4\pi_3^r & \leq & 0, \\ \pi_1^r & & & \leq & 1, \\ & \pi_2^r & & \leq & 1, \\ & & \pi_3^r & \leq & 1. \end{cases}$$

Similarly to the previous iteration, we have $(\pi_1^r, \pi_2^r, \pi_3^r) = (1,1,1) - (7/3, 5/3, 0) = (-4/3, -2/3, 1)$. To find the maximum θ such that the vector $\widetilde{\pi} = (\frac{19}{42}, \frac{-5}{14}, \frac{5}{42})^T + \theta(-4/3, -2/3, 1)^T$ is feasible in **D**, we plug this $\widetilde{\pi}$ into all inequalities of **D**. From the first inequality we get

$$\widetilde{\pi}^T \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix} = (\frac{19}{42}, \frac{-5}{14}, \frac{5}{42}) \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix} + \theta(-4/3, -2/3, 1) \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix} = 1 + \theta \cdot (-4 - 2 + 6) = 1 \leq 1,$$

which holds for every θ. Similarly, from the second inequality of **D** we get $3 + \theta(-16/3 + 4/3 + 4) \leq 3$ which also holds for every θ. From the third inequality of **D** we get $-7/2 + \theta(4 - 4 + 0) \leq 3$ which holds for every θ. Finally, from the fourth inequality of **D** we get $13/14 + \theta(-4/3 + 2/3 + 1) \leq 1$ which holds for $\theta \leq 3/14$.

Thus we choose $\theta = 3/14$ and hence our new π is $(\frac{19}{42}, \frac{-5}{14}, \frac{5}{42})^T + \frac{3}{14}(-4/3, -2/3, 1)^T = (\frac{1}{6}, \frac{-1}{2}, \frac{1}{3})^T$. Note that now $w = 2\frac{1}{6} - \frac{1}{2} + 4\frac{1}{3} = \frac{7}{6}$.

We start our cycle again. Now $J = \{1, 2, 4\}$. Thus, our new restricted primal **RP3** is

$$\text{Minimize } \xi = x_1^r + x_2^r + x_3^r$$

subject to

$$\begin{cases} 3x_1 & +4x_2 & +x_4 & +x_1^r & & & = & 2, \\ 3x_1 & -2x_2 & -x_4 & & +x_2^r & & = & 1, \\ 6x_1 & +4x_2 & +x_4 & & & +x_3^r & = & 4, \\ x_1, & x_2, & x_4, & x_1^r, & x_2^r, & x_3^r & \geq & 0. \end{cases}$$

We use the modified last tableau

		x_1	x_2	x_4	x_1^r	x_2^r	x_3^r
$y_0 = -\xi$	$-2/3$	0	0	$-1/3$	$7/3$	$5/3$	0
x_2	$1/6$	0	1	$1/3$	$1/6$	$-1/6$	0
x_1	$4/9$	1	0	$-1/9$	$1/9$	$2/9$	0
x_3^r	$2/3$	0	0	$1/3$	$-4/3$	$-2/3$	1

where the third column was obtained using the formulas

$$\widetilde{c}_4 = c_4 - (-4/3, -2/3, 1)\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0 - 1/3 = -1/3,$$

and $\widetilde{A}_4 = B^{-1}A_4 = \begin{pmatrix} 1/6 & -1/6 & 0 \\ 1/9 & 2/9 & 0 \\ -4/3 & -2/3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ -1/9 \\ 1/3 \end{pmatrix}$.

We pivot on x_4-column and Row 1. The result is

		x_1	x_2	x_4	x_1^r	x_2^r	x_3^r
$y_0 = -\xi$	$-1/2$	0	1	0	$5/2$	$3/2$	0
x_4	$1/2$	0	3	1	$1/2$	$-1/2$	0
x_1	$1/2$	1	$1/3$	0	$1/6$	$1/6$	0
x_3^r	$1/2$	0	-1	0	$-3/2$	$-1/2$	1

As above, the optimal vector of the new restricted dual **DRP3** is $(\pi_1^r, \pi_2^r, \pi_3^r)^T = (1, 1, 1)^T - (5/2, 3/2, 0)^T = (-3/2, -1/2, 1)^T$. To find the maximum θ such that the vector $\widetilde{\pi} = (\frac{1}{6}, \frac{-1}{2}, \frac{1}{3})^T + \theta(-3/2, -1/2, 1)^T$ is feasible in **D**, we do not need to check inequalities in **D** corresponding to x_1 and x_4, since they are in the basis of **RP3**. From the remaining two inequalities we get

$$\widetilde{\pi}^T\begin{pmatrix} -3 \\ 6 \\ 0 \end{pmatrix} = (\frac{1}{6}, \frac{-1}{2}, \frac{1}{3})\begin{pmatrix} -3 \\ 6 \\ 0 \end{pmatrix} + \theta(-3/2, -1/2, 1)\begin{pmatrix} -3 \\ 6 \\ 0 \end{pmatrix} = -7/2 + \theta \cdot (9/2 - 3 + 0) \leq 3,$$

which holds for $\theta \leq 13/3$, and $3 + \theta(-6 + 1 + 4) \leq 3$ which holds for each positive θ.

Thus we choose $\theta = 13/3$ and hence our new π is $(\frac{1}{6}, \frac{-1}{2}, \frac{1}{3})^T + \frac{13}{3}(-3/2, -1/2, 1)^T ==$ $(-\frac{19}{3}, \frac{-8}{3}, \frac{14}{3})^T$. Note that now $w = -2\frac{19}{3} - \frac{8}{3} + 4\frac{14}{3} = \frac{10}{3}$.

We start our cycle again. Now $J = \{1, 3, 4\}$. Note that 2 is not in J anymore. The tableau corresponding to the new restricted primal **RP4** is

		x_1	x_3	x_4	x_1^r	x_2^r	x_3^r
$y_0 = -\xi$	$-1/2$	0	$-3/2$	0	$5/2$	$3/2$	0
x_4	$1/2$	0	$-9/2$	1	$1/2$	$-1/2$	0
x_1	$1/2$	1	$1/2$	0	$1/6$	$1/6$	0
x_3^r	$1/2$	0	$3/2$	0	$-3/2$	$-1/2$	1

We got the column for x_3 using the formulas

$$\widetilde{c}_3 = c_3 - (-3/2, -1/2, 1)\begin{pmatrix} -3 \\ 6 \\ 0 \end{pmatrix} = 0 - 9/2 + 6/2 = -3/2,$$

and $\widetilde{A}_3 = B^{-1}A_3 = \begin{pmatrix} 1/2 & -1/2 & 0 \\ 1/6 & 1/6 & 0 \\ -3/2 & -1/2 & 1 \end{pmatrix}\begin{pmatrix} -3 \\ 6 \\ 0 \end{pmatrix} = \begin{pmatrix} -9/2 \\ 1/2 \\ 3/2 \end{pmatrix}.$

Pivoting on x_3-column and Row 3 we get

		x_1	x_3	x_4	x_1^r	x_2^r	x_3^r
$y_0 = -\xi$	0	0	0	0	1	1	1
x_4	2	0	0	1	-4	-2	3
x_1	$1/3$	1	0	0	$2/3$	$1/3$	$-1/3$
x_3	$1/3$	0	1	0	-1	$-1/3$	$2/3$

So, vector $(-\frac{19}{3}, \frac{-8}{3}, \frac{14}{3})^T$ indeed is an optimal vector in **D** and the corresponding optimal vector in **P** is $(1/3, 0, 1/3, 2)^T$. The optimal cost is $10/3$.