

## An example of the dual simplex method

Suppose we are given the problem

$$\begin{aligned} & \text{Minimize } z = 2x_1 + 3x_2 + 4x_3 + 5x_4 \\ & \text{subject to } \begin{cases} x_1 - x_2 + x_3 - x_4 \geq 10, \\ x_1 - 2x_2 + 3x_3 - 4x_4 \geq 6, \\ 3x_1 - 4x_2 + 5x_3 - 6x_4 \geq 15 \\ x_1, x_2, x_3, x_4 \geq 0. \end{cases} \end{aligned} \tag{1}$$

If we would have inequalities  $\leq$  instead of  $\geq$ , then the usual simplex would work nicely. The two-phase method is more tedious. But since all coefficients in  $z = 2x_1 + 3x_2 + 4x_3 + 5x_4$  are non-negative, we are fine for the dual simplex.

Multiply the equations by  $-1$  and add to each of the equations its own variable. Then we get the following tableau.

		$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$
$x_0 = -z$	0	2	3	4	5	0	0	0
$x_5$	-10	-1	1	-1	1	1	0	0
$x_6$	-6	-1	2	-3	4	0	1	0
$x_7$	-15	-3	4	-5	6	0	0	1

Choose Row 1 to pivot on. The ratio for  $x_1$  ( $\frac{2}{-1} = -2$ ) is greater than the ratio for  $x_3$  ( $\frac{4}{-1} = -4$ ), so pivot on  $a_{1,1}$ . After pivoting, we get

		$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$
$x_0 = -z$	-20	0	5	2	7	2	0	0
$x_1$	10	1	-1	1	-1	-1	0	0
$x_6$	4	0	1	-2	3	-1	1	0
$x_7$	15	0	1	-2	3	-3	0	1

Now every  $a_{i,0}$  for  $i > 0$  is nonnegative. So, the tableau is optimal. But suppose that the boss adds now the new restriction:

$$x_1 + 2x_2 + 3x_3 - 4x_4 \leq 8.$$

With the dual simplex, we do not need to start from scratch. We simply add the new row and one more column to our final tableau.

		$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$
$x_0 = -z$	-20	0	5	2	7	2	0	0	0
$x_1$	10	1	-1	1	-1	-1	0	0	0
$x_6$	4	0	1	-2	3	-1	1	0	0
$x_7$	15	0	1	-2	3	-3	0	1	0
$x_8$	8	1	2	3	-4	0	0	0	1

Excluding from the last row  $x_1, x_6$  and  $x_7$ , we get the tableau

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$
$x_0 = -z$	-20	0	5	2	7	2	0	0
$x_1$	10	1	-1	1	-1	-1	0	0
$x_6$	4	0	1	-2	3	-1	1	0
$x_7$	15	0	1	-2	3	-3	0	1
$x_8$	-2	0	3	2	-3	1	0	0

Note that if in the last row there were no  $-3$ , then the LP would be infeasible. Now we pivot on  $x_4$ :

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$
$x_0 = -z$	$-74/3$	0	12	$20/3$	0	$13/3$	0	$7/3$
$x_1$	$32/3$	1	-2	$1/3$	0	$-4/3$	0	$-1/3$
$x_6$	2	0	4	0	0	0	1	0
$x_7$	13	0	4	0	0	-2	0	1
$x_4$	$2/3$	0	-1	$-2/3$	1	$-1/3$	0	$-1/3$

Thus we got a solution again.