# Second example of simplex method

Suppose we are given the problem

$$\text{Minimize } z = -x_1 + x_2 - x_3$$

subject to

$$\left\{\begin{array}{rrrrrrcr} 2x_1 & -x_2 & +2x_3 & +x_4 & & & = & 4 \\ 2x_1 & -3x_2 & +x_3 & & +x_5 & & = & -5 \\ -x_1 & +x_2 & -2x_3 & & & +x_6 & = & -1 \\ x_1, & x_2, & x_3, & x_4, & x_5 & x_6 & \geq & 0. \end{array}\right. \tag{1}$$

This system is solved with respect to $x_4, x_5$, and $x_6$, but the obtained basic solution is not feasible. So, we will look for a feasible solution by solving another linear program obtained as follows.

Multiply the last two equations by $-1$ in order to get positive RHS, then add to either of these equations its own variable and switch the LHS with the RHS:

$$\left\{\begin{array}{rcrrrrrrrrcr} 4 & = & 2x_1 & -x_2 & +2x_3 & +x_4 & & & & & & \\ 5 & = & -2x_1 & +3x_2 & -x_3 & & -x_5 & & +y_1 & & & \\ 1 & = & x_1 & -x_2 & +2x_3 & & & -x_6 & & +y_2 & & \\ x_1, & x_2, & x_3, & x_4, & x_5, & x_6, & y_1, & y_2 & & \geq & 0. \end{array}\right. \tag{2}$$

Note that a basic feasible solution of system (2) with $y_1 = y_2 = 0$ would be a basic feasible solution of (1). So, in search of such solutions, we will attempt to minimize $\xi = y_1 + y_2$ under conditions (2). A good feature is that we already have the following basic feasible solution of (2): $x_1 = x_2 = x_3 = x_5 = x_6 = 0$, $x_4 = 4$, $y_1 = 5$, $y_2 = 1$. Consider the tableau corresponding to our new linear program:

| | | $y_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $y_1$ | $y_2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $y_0 = -\xi$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| $x_4$ | 4 | 0 | 2 | $-1$ | 2 | 1 | 0 | 0 | 0 | 0 |
| $y_1$ | 5 | 0 | $-2$ | 3 | $-1$ | 0 | $-1$ | 0 | 1 | 0 |
| $y_2$ | 1 | 0 | 1 | $-1$ | 2 | 0 | 0 | $-1$ | 0 | 1 |

We cannot yet start pivoting, since the coefficients at basic variables $y_1$ and $y_2$ in Row 0 are non-zeros. Excluding $y_1$ and $y_2$ from Row 0, we get

| | | $y_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $y_1$ | $y_2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $y_0 = -\xi$ | $-6$ | 1 | 1 | $-2$ | $-1$ | 0 | 1 | 1 | 0 | 0 |
| $x_4$ | 4 | 0 | 2 | $-1$ | 2 | 1 | 0 | 0 | 0 | 0 |
| $y_1$ | 5 | 0 | $-2$ | 3 | $-1$ | 0 | $-1$ | 0 | 1 | 0 |
| $y_2$ | 1 | 0 | 1 | $-1$ | 2 | 0 | 0 | $-1$ | 0 | 1 |

Choose Column $x_3$ as pivot column. Then the pivot row will be Row 3:

| | | $y_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $y_1$ | $y_2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $y_0 = -\xi$ | $-11/2$ | 1 | $3/2$ | $-5/2$ | 0 | 0 | 1 | $1/2$ | 0 | $1/2$ |
| $x_4$ | 3 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | $-1$ |
| $y_1$ | $11/2$ | 0 | $-3/2$ | $5/2$ | 0 | 0 | $-1$ | $-1/2$ | 1 | $1/2$ |
| $x_3$ | $1/2$ | 0 | $1/2$ | $-1/2$ | 1 | 0 | 0 | $-1/2$ | 0 | $1/2$ |

Now we pivot on $x_2$:

| | | $y_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $y_1$ | $y_2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $y_0 = -\xi$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| $x_4$ | 3 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | $-1$ |
| $x_2$ | $11/5$ | 0 | $-3/5$ | 1 | 0 | 0 | $-2/5$ | $-1/5$ | $2/5$ | $1/5$ |
| $x_3$ | $16/10$ | 0 | $2/10$ | 0 | 1 | 0 | $-1/5$ | $-6/10$ | $1/5$ | $6/10$ |

Thus we found a basic feasible solution of (1) and return to this problem: Delete the columns corresponding to $y_1$ and $y_2$, and replace the objective function.

| | | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|---|---|
| $x_0 = -z$ | 0 | 1 | $-1$ | 1 | $-1$ | 0 | 0 | 0 |
| $x_4$ | 3 | 0 | 1 | 0 | 0 | 1 | 0 | 1 |
| $x_2$ | $11/5$ | 0 | $-3/5$ | 1 | 0 | 0 | $-2/5$ | $-1/5$ |
| $x_3$ | $16/10$ | 0 | $2/10$ | 0 | 1 | 0 | $-1/5$ | $-6/10$ |

Excluding basic variables $x_2$ and $x_3$ from Row 0, we get

| | | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|---|---|
| $x_0 = -z$ | $-3/5$ | 1 | $-1/5$ | 0 | 0 | 0 | $1/5$ | $-2/5$ |
| $x_4$ | 3 | 0 | 1 | 0 | 0 | 1 | 0 | 1 |
| $x_2$ | $11/5$ | 0 | $-3/5$ | 1 | 0 | 0 | $-2/5$ | $-1/5$ |
| $x_3$ | $16/10$ | 0 | $2/10$ | 0 | 1 | 0 | $-1/5$ | $-6/10$ |

Choose $x_6$ as the pivot column. Then the pivot row is Row 1. After the pivot we have

| | | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|---|---|
| $x_0 = -z$ | $3/5$ | 1 | $1/5$ | 0 | 0 | $2/5$ | $1/5$ | 0 |
| $x_6$ | 3 | 0 | 1 | 0 | 0 | 1 | 0 | 1 |
| $x_2$ | $14/5$ | 0 | $-2/5$ | 1 | 0 | $1/5$ | $-2/5$ | 0 |
| $x_3$ | $17/5$ | 0 | $4/5$ | 0 | 1 | $3/5$ | $-1/5$ | 0 |

This tableau corresponds to the basic solution $x_5 = x_1 = x_4 = 0$, $x_2 = 14/5$, $x_3 = 17/5$, $x_6 = 3$, which gives $-z = 3/5$. Since we do not have negative entries in Row 0, this solution is optimal.