First example of the simplex method

Suppose we are given the problem

$$\text{Minimize } z = -x_1 + 2x_2 - x_3$$

subject to
$$\begin{cases} x_1 & -2x_2 & +x_3 & +x_4 & & = & 10 \\ 2x_1 & -3x_2 & -x_3 & & +x_5 & = & 6 \\ x_1, & x_2, & x_3, & x_4, & x_5 & \geq & 0. \end{cases}$$

Rewrite the objective function as a new equation with the new variable z and switch the LHS with the RHS.

$$\begin{cases} 0 & = & -z & -x_1 & +2x_2 & -x_3 & & \\ 10 & = & & x_1 & -2x_2 & +x_3 & +x_4 & \\ 6 & = & & 2x_1 & -3x_2 & -x_3 & & +x_5 \\ x_1, & x_2, & x_3, & x_4, & x_5 & \geq & 0. & \end{cases}$$

A good thing about this system is that it is solved with respect to x_4 and x_5. Introduce the new variable $x_0 = -z$ and consider the tableau consisting of the coefficients of the system:

		x_0	$\mathbf{x_1}$	x_2	x_3	x_4	x_5
$x_0 = -z$	0	1	-1	2	-1	0	0
x_4	10	0	1	-2	1	1	0
x_5	6	0	**2**	-3	-1	0	1

The variables on the left indicate with respect to which variable the corresponding equation is solved. Note that this tableau corresponds to the basic solution $x_1 = x_2 = x_3 = 0$, $x_4 = 10$, $x_5 = 6$ which gives $-z = 0$.

Check the 0th row whether there are negative entries in columns distinct from 0th. There are negative coefficients at x_1 and x_3. Choose, for example x_1 (it is in bold in the picture). We call column 1 the *pivot column*. Both entries in this column are positive. The ratio 10/1 is greater than the ratio 6/2, so the *pivot row* will be Row 2, and the entry in row 2 and column 1 will be the *pivot entry*. Pivoting (excluding x_1 from all other equations), we come to the tableau

		x_0	x_1	x_2	x_3	x_4	x_5
$x_0 = -z$	3	1	0	$1/2$	$-3/2$	0	$1/2$
x_4	7	0	0	$-1/2$	**3/2**	1	$-1/2$
x_1	3	0	1	$-3/2$	$-1/2$	0	$1/2$

This tableau corresponds to the basic solution $x_5 = x_2 = x_3 = 0$, $x_1 = 3$, $x_4 = 7$ which gives $-z = 3$. So, this solution is better than the previous. Now, the only negative entry in the 0th row is the coefficient at x_3. So, this is the pivot column. In this column, there is only one positive entry, so we pivot on this entry:

		x_0	x_1	x_2	x_3	x_4	x_5
$x_0 = -z$	10	1	0	0	0	1	0
x_3	14/3	0	0	$-1/3$	1	2/3	$-1/3$
x_1	16/3	0	1	$-5/3$	0	1/3	1/3

This tableau corresponds to the basic solution $x_5 = x_2 = x_4 = 0$, $x_1 = 16/3$, $x_3 = 14/3$, which gives $-z = 10$. This solution not only is better than the previous, but since we do not have negative entries in Row 0, this solution is optimal (!!). Indeed, the 0th row reads as the following equation:

$$z = -10 + x_4.$$

Since $x_4 \geq 0$, z never can be less than -10 and we are given a solution with $z = -10$.