Bland's algorithm does not cycle

Suppose we are at a step of the simplex procedure represented by tableau T. Let $a_{i,j}$ be the entry at row i and column j of T. Suppose that there exists $1 \le i \le n$ such that $a_{0,i} < 0$, since otherwise we would have found a optimal basic feasible solution. Let $j_1, \dots, j_m$ be such that row i is solved for j_i, i.e. $a_{j,j_i} = 1$ if $j = i$ and $a_{j,j_i} = 0$ if $j \neq j_i$.for $j \in \{0, \dots, m\}$. Bland's rules for selecting a pivot column s is to pick s so that $s = \min\{j : a_{0,j} < 0, 1 \le j \le n\}$. If the $a_{r,s} \le 0$ for every $r \in \{1, \dots, m\}$ then the LP is unbounded, so assume this is not the case. Choose a pivot row r so that among all $r \in \{1, \dots, m\}$ such that $a_{r,s} > 0$ and $a_{r,0}/a_{r,s} = \min\{a_{j,0}/a_{j,s} : a_{j,s} > 0\}$, j_r is minimum.

Sketch of proof that Bland's algorithm does not cycle.

Assume that while using Bland's algorithm we encounter tableaus $T_1, \dots, T_\alpha$ in consecutive steps such that $T_1 = T_\alpha$. Let $x_{l_1}, \dots, x_{l_k}$ with $l_1 < l_2 < \cdots < l_k$ be the variables which enter and leave the basis during these α steps. Note that the value in row 0 column 0 never decreases in any step during the simplex procedure. This implies that every step in the cycle is degenerate. Furthermore, this implies that the bfs does not change throughout the cycle and $x_{l_i} = 0$ in every bfs associated with the tableaus $T_1, \dots, T_\alpha$. Let T be the tableau before x_{l_k} enter the basis and let $\widetilde{T}$ be the table before x_{l_k} leaves the basis. Let $a_{i,j}$ be row i and column j of tableau T and Let $\widetilde{a}_{i,j}$ be row i and column j of tableau $\widetilde{T}$.

Suppose $\{j_1, \dots, j_m\}$ are such that row i is solved for x_{j_i} in in $\widetilde{T}$. Since x_{l_k} is about to leave the basis, there exists u such that $j_u = l_k$, i.e. row u is solved for x_{l_k} in $\widetilde{T}$. Suppose x_{l_s} enters the basis on the step associated with $\widetilde{T}$. This implies l_s is the pivot column and u is the pivot row. Let $i \in \{1 \dots, m\}$ be such that $j_i \in \{l_1, \dots, l_{k-1}\}$. In the bfs associated with $\widetilde{T}$, $x_{l_k} = 0$, and $x_{j_i} = 0$. So, since row i is solved for j_i and row u is solved for l_k, $\widetilde{a}_{i,0} = 0$ and $\widetilde{a}_{u,0} = 0$. This implies that, because of Bland's rule for selecting the pivot row, we must have $\widetilde{a}_{i,l_s} \le 0$.

For $j \in \{1, \dots n\}$ define

$$y_j = \begin{cases} 1 & \text{if } j = l_s, \\ a_{i,0} - a_{i,l_s} & \text{if } j = j_i, \\ 0 & \text{otherwise.} \end{cases}$$

Note the following facts about the vector $y = (y_1, \dots, y_n)^T$.

(1) For every $j \in \{l_1, \dots, l_{k-1}\}$, $y_j \ge 0$ and $y_{l_k} < 0$:
From the previous discussion, we have that for every $i \in \{1 \dots, m\}$ such that $j_i \in \{l_1, \dots, l_{k-1}\}$, $a_{i,0} = 0$ and $a_{i,l_s} \le 0$, so $y_{j_i} \ge 0$. Since for $j \in \{l_1, \dots, l_{k-1}\} \setminus \{j_1, \dots, j_m\}$, $y_j = 0$, we have $y_j \ge 0$ for every $j \in \{l_1, \dots, l_{k-1}\}$. Also, $y_{l_k} = a_{u,0} - a_{u,l_s} = -a_{u,l_s} < 0$.

(2) For every j_i, $\widetilde{a}_{0,j_i} = 0$, so $\sum_{j=1}^{n} \widetilde{a}_{0,j} y_i = \widetilde{a}_{0,l_s}$. Since l_s was selected as the pivot row, $\widetilde{a}_{0,l_s} < 0$, which implies $\sum_{j=1}^{n} \widetilde{a}_{0,j} y_j < 0$.

(3) The vector y need not be basic or feasible, but, using the tableau $\widetilde{T}$, it is not hard to check that $Ay = b$. Therefore, row 0 of the tableau gives us that $\widetilde{a}_{0,0} =$

$-c^T y + \sum_{j=1}^{n} \widetilde{a}_{0,j} y_j$. Therefore,

$$c^T y = -\widetilde{a}_{0,0} + \sum_{j=1}^{n} \widetilde{a}_{0,j} y_j < -\widetilde{a}_{0,0}.$$

Now consider the tableau T. Since x_{l_k} is about enter the basis, column l_k will be selecting as the pivot column by Bland's algorithm. This implies $a_{0,l_i} \geq 0$ for every $i \in \{1, \ldots, k-1\}$ and $a_{0,l_k} < 0$. Again since $Ay = b$, row 0 of T gives us that $c^T y = -a_{0,0} + \sum_{j=1}^{n} a_{0,j} y_j$. If x_j is in the basis associated with T, then $a_{0,j} = 0$. If x_j is not in the basis associated with T and $j \notin \{l_1, \ldots, l_k\}$, then $y_j = 0$, because x_j is not in the basis associated with $\widetilde{T}$. When $j \in \{l_1, \ldots, l_{k-1}\}$ we have that $a_{0,j} \geq 0$ and, by (1), $y_j \geq 0$. Also, by (1), $y_{l_k} < 0$, so $a_{0,l_k} y_{l_k} > 0$. Therefore, $\sum_{j=1}^{n} a_{0,j} y_j > 0$. This implies $c^T y > -a_{0,0} = -\widetilde{a}_{0,0}$ and this contradicts (3).