Math-484 Homework #6 (KKT conditions)

Due 10am Mar 17
Write your name on your solutions and indicate if you are a C14 (4 credit hour) student.

1: Prove that if M is a subspace of $\mathbb{R}^n$ such that $M \neq \mathbb{R}^n$, then the interior of M is empty.

2: Let A be an $m \times n$-matrix and let $C = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n \text{ and } \mathbf{x} \geq \mathbf{0}\} \subseteq \mathbb{R}^m$. Prove that if there exists $\mathbf{a} \in \mathbb{R}^m$ such that $\mathbf{a} \cdot \mathbf{c} \leq \alpha$ for all $\mathbf{c} \in C$, then $\mathbf{a} \cdot \mathbf{c} \leq 0$ for all $\mathbf{c} \in C$.

3: Let A be an $m \times n$-matrix, with columns $A_1, \ldots, A_n \in \mathbb{R}^m$ and let $\mathbf{b} \in \mathbb{R}^m$. You can assume that the set

$$C = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n \text{ and } \mathbf{x} \geq \mathbf{0}\} \subseteq \mathbb{R}^m$$

is closed and convex. Show that if there does not exist $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$, then there exists $\mathbf{a} \in \mathbb{R}^m$ such that

$$\mathbf{a} \cdot \mathbf{b} > 0 \text{ and } \mathbf{a} \cdot A_j \leq 0 \text{ for every } j \in \{1, \ldots, n\}.$$

Hint: Use the previous problem and the basic separation theorem with the set C and the vector $\mathbf{b}$. Note that this is a version of Farkas Lemma from linear programming.

4: Let (P) be a convex program and let

$$S := \{x \in C : f(x) = MP \text{ and } g_i(x) \leq 0 \text{ for all } i \in [m]\ \}.$$

Prove that S is a convex set.
Hint: First show that if f is convex on C, then for any $M \in \mathbb{R}$, the set $\{x \in C : f(x) \leq M\}$ is convex.

5: Question #5 has been removed, because we haven't covered the necessary material. The question that was here will be asked on homework #7.

6: *(C14 only)* Let A be an $m \times n$ matrix and let $\mathbf{b} \in \mathbb{R}^m$ be a fixed vector. Suppose that the convex program

$$(P) \begin{cases} \text{Minimize} & \|\mathbf{x}\|^2 \\ \text{subject to} & A\mathbf{x} \leq \mathbf{b} \end{cases}$$

is superconsistent and has a solution $\mathbf{x}^*$. Use Karush-Kuhn-Tucker Theorem to show that there is a vector $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{x}^* = A^T\mathbf{y}$.