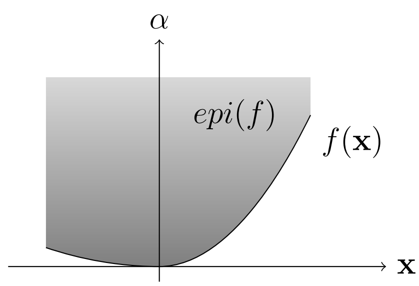
Math-484 Homework #4 (convex functions and $(A-G)$ inequality)

Due 10am Feb 24.
Write your name on your solutions and indicate if you are a C14 (4 credit hour) student.

1: Let $D \subset \mathbb{R}^n$ be convex and $f : D \to \mathbb{R}$. The *epigraph* of f is a subset of $\mathbb{R}^{n+1}$ defined by

$$epi(f) = \{(\mathbf{x}, \alpha) : \mathbf{x} \in D, \alpha \in \mathbb{R}, f(\mathbf{x}) \le \alpha\}.$$



Intuitively, *epigraph* are vectors above the graph of f including the graph of f.
a) Sketch the epigraph of the function $f(x) = e^x$ for $x \in \mathbb{R}$.
b) Show that $f(\mathbf{x})$ is convex if and only if epi(f) is convex.
c) Show that if $f(\mathbf{x})$ and $g(\mathbf{x})$ are convex functions defined on a convex set C then

$$h(\mathbf{x}) := \max\{f(\mathbf{x}), g(\mathbf{x})\}$$

is also a convex function on C by showing that

$$\text{epi}(\max\{f(\mathbf{x}), g(\mathbf{x})\}) = \text{epi}(f(\mathbf{x})) \cap \text{epi}(g(\mathbf{x})).$$

2: Prove the following statement:
If $f(\mathbf{x}) : C \to \mathbb{R}$ is a concave function and $g(y)$ be an increasing concave function defined on the range of $f(\mathbf{x})$ then $g(f(\mathbf{x}))$ is a concave function.
Hint: See Theorem 2.3.10 (c) and its proof.

3: Use the Arithmetic-Geometric Mean inequality to find the smallest radius r such that a circular cylinder of volume 8 cubic units can be inscribed in the sphere of radius r.
Note that you must use the Arithmetic-Geometric Mean inequality in your solution.

4: Solve using $(A-G)$ inequality the following problems: (Get the value of objective function and compute (x^*, y^*, z^*))
a) Minimize $x^2 + y + z$ subject to $xyz = 1$ and $x, y, z > 0$
b) Maximize xyz subject to $3x + 4y + 12z = 1$ and $x, y, z > 0$
c) Minimize $3x + 4y + 12z$ subject to $xyz = 1$ and $x, y, z > 0$

5: Show that for all positive x and y:

$$\frac{x}{4} + \frac{3y}{4} \leq \sqrt{\ln\left(\frac{e^{x^2}}{4} + \frac{3}{4}e^{y^2}\right)}$$

Hint: Use the fact that if $a \leq b$ and f is increasing, then $f(a) \leq f(b)$. The desired inequality will follow from the convexity of an appropriately chosen function.

6: *(D14 only)* Let $\mathcal{F}$ be the set of all functions $f : \mathbb{R} \to \mathbb{R}$ where $f(x) > 0$ for every $x \in \mathbb{R}$. Let

$$\|f - g\| = \sup_{x \in \mathbb{R}} |f(x) - g(x)|.$$

Determine if $\mathcal{F}$ is convex, open and/or closed.