Two phase simplex

- Suppose we are given the problem

$$\text{Minimize } z = -x_1 + x_2 - x_3$$

subject to

$$\begin{array}{ccccccccc} 2x_1 & -x_2 & +2x_3 & +x_4 & & & = & 4 \\ 2x_1 & -3x_2 & +x_3 & & +x_5 & & = & -5 \\ -x_1 & +x_2 & -2x_3 & & & +x_6 & = & -1 \\ x_1, & x_2, & x_3, & x_4, & x_5 & x_6 & \geq & 0. \end{array} \qquad (1)$$

- This system is solved with respect to x_4, x_5, and x_6, but the obtained basic solution is not feasible. So, we will look for a feasible solution by solving another linear program obtained as follows.

- Multiply the last two equations by -1 in order to get positive RHS, then add to either of these equations its own variable and switch the LHS with the RHS:

$$\begin{array}{cccccccccc} 4 & = & 2x_1 & -x_2 & +2x_3 & +x_4 & & & & \\ 5 & = & -2x_1 & +3x_2 & -x_3 & & -x_5 & & +x_7 & \\ 1 & = & x_1 & -x_2 & +2x_3 & & & -x_6 & & +x_8 \\ & & x_1, & x_2, & x_3, & x_4, & x_5, & x_6, & x_7, & x_8 \geq 0. \end{array} \qquad (2)$$

- Note that a basic feasible solution of system (2) with $x_7 = x_8 = 0$ would be a basic feasible solution of (3).
- So, in search of such solutions, we will attempt to minimize $\xi = x_7 + x_8$ under conditions (2).
- We have added x_7 and x_8 so that we have the following basic feasible solution of (2): $x_1 = x_2 = x_3 = x_5 = x_6 = 0$, $x_4 = 4$, $x_7 = 5$, $x_8 = 1$.
- We could also add a variable for the first row, but we don't have to since it is already solved for x_4.

- Consider the tableau corresponding to our new linear program:

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
$-\xi$	0	0	0	0	0	0	0	1	1
x_4	4	2	-1	2	1	0	0	0	0
x_7	5	-2	3	-1	0	-1	0	1	0
x_8	1	1	-1	2	0	0	-1	0	1

- We cannot yet start pivoting, since the coefficients at basic variables x_7 and x_8 in Row 0 are non-zeros. Excluding x_7 and x_8 from Row 0, we get...

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
$-\xi$	-6	1	-2	-1	0	1	1	0	0
x_4	4	2	-1	2	1	0	0	0	0
x_7	5	-2	3	-1	0	-1	0	1	0
x_8	1	1	-1	2	0	0	-1	0	1

Choose Column x_3 as the pivot column. Then the pivot row will be Row 3:

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
$-\xi$	$-11/2$	$3/2$	$-5/2$	0	0	1	$1/2$	0	$1/2$
x_4	3	1	0	0	1	0	1	0	-1
x_7	$11/2$	$-3/2$	$5/2$	0	0	-1	$-1/2$	1	$1/2$
x_3	$1/2$	$1/2$	$-1/2$	1	0	0	$-1/2$	0	$1/2$

Now we pivot on column x_2 and Row 2:

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
$-\xi$	0	0	0	0	0	0	0	1	1
x_4	3	1	0	0	1	0	1	1	-1
x_2	$11/5$	$-3/5$	1	0	0	$-2/5$	$-1/5$	$2/5$	$1/5$
x_3	$16/10$	$2/10$	0	1	0	$-1/5$	$-6/10$	$1/5$	$6/10$

- This is an optimal tableau for the auxiliary problem
- If the value of the objective function at the optimum was greater than 0, then ...
- we can could conclude that the original problem was infeasible.
- But it is 0, so we have found a bfs of the original problem
- Delete the columns corresponding to x_7 and x_8, and replace the original objective function.
- One special case that could happen:
  - If the optimum to the auxiliary problem is degenerate, an artificial variable, x_7 or x_8, could still be in the basis.
  - To *drive the artificial variables out of the basis* if row i is solved for, say x_7 (so x_7 is still in the basis), then pivot on entry (i, j) for any of the original columns j such that $a_{i,j} \neq 0$ (even if $\bar{c}_j > 0$ or $a_{i,j} < 0$).
  - Repeat until no artifical variables remain in the basis. (See page 56 of the book).

- Here is the new tableau - the top row corresponds exactly to the original objective function

		x_1	x_2	x_3	x_4	x_5	x_6
$-z$	0	-1	1	-1	0	0	0
x_4	3	1	0	0	1	0	1
x_2	$11/5$	$-3/5$	1	0	0	$-2/5$	$-1/5$
x_3	$16/10$	$2/10$	0	1	0	$-1/5$	$-6/10$

- Excluding basic variables x_2 and x_3 from Row 0, we get

		x_1	x_2	x_3	x_4	x_5	x_6
$-z$	$-3/5$	$-1/5$	0	0	0	$1/5$	$-2/5$
x_4	3	1	0	0	1	0	1
x_2	$11/5$	$-3/5$	1	0	0	$-2/5$	$-1/5$
x_3	$16/10$	$2/10$	0	1	0	$-1/5$	$-6/10$

- Choose x_6 as the pivot column. Then the pivot row is Row 1. After the pivot we have

		x_1	x_2	x_3	x_4	x_5	x_6
$-z$	$3/5$	$1/5$	0	0	$2/5$	$1/5$	0
x_6	3	1	0	0	1	0	1
x_2	$14/5$	$-2/5$	1	0	$1/5$	$-2/5$	0
x_3	$17/5$	$4/5$	0	1	$3/5$	$-1/5$	0

This tableau corresponds to the basic solution $x_5 = x_1 = x_4 = 0$, $x_2 = 14/5$, $x_3 = 17/5$, $x_6 = 3$, which gives $-z = 3/5$. Since we do not have negative entries in Row 0, this solution is optimal.

Example 2

$$\text{Minimize } z = x_1 - x_2 - x_3$$

subject to

$$\begin{array}{rrrcl} 2x_1 & +4x_2 & +4x_3 & = & 4 \\ 3x_1 & -x_2 & -2x_3 & = & 6 \\ x_1, & x_2, & x_3 & \geq & 0. \end{array} \tag{3}$$

- In general, it is hard to find a feasible basis, but for this small example it is not too difficult.
- Both $(1, 2)$ and $(1, 3)$ are feasible bases, with bfs $(2, 0, 0)^T$. Suppose you didn't notice this, so you do the first phase of two phase simplex.
- Add artificial variables x_4 and x_5 and use the objective function $\xi = x_4 + x_5$.

		x_1	x_2	x_3	x_4	x_5
$-\xi$	0	0	0	0	1	1
x_4	4	2	4	4	1	0
x_5	6	3	-1	-2	0	1

Before we start simplex, we exclude the basic variables x_4 and x_5 from the top row. Which gives

		x_1	x_2	x_3	x_4	x_5
$-\xi$	-10	-5	-3	-2	0	0
x_4	4	2	4	4	1	0
x_5	6	3	-1	-2	0	1

We following Bland's rule which gives pivot column x_1 and pivot row 1. Pivoting gives

		x_1	x_2	x_3	x_4	x_5
$-\xi$	0	0	7	8	$5/2$	0
x_1	2	1	2	2	$1/2$	0
x_5	0	0	-7	-8	$-3/2$	1

We seem to be done with the first phase and $\xi = 0 = x_4 = x_5$ is zero in the current bfs $(2, 0, 0, 0, 0)^T$, so $(2, 0, 0)^T$ is a bfs for the original problem.

		x_1	x_2	x_3	x_4	x_5
$-\xi$	0	0	7	8	$5/2$	0
x_1	2	1	2	2	$1/2$	0
x_5	0	0	-7	-8	$-3/2$	1

The issue here is that the artifical variable x_5 is still in the basis. To move x_5 out of the basis, we only have to pivot on some non-zero entry in the row 2 that is in a column corresponding to a variable that is not artificial. We could pick either entry $a_{2,2}$ or $a_{2,3}$. We choose $a_{2,2}$ arbitrarily, which gives.

		x_1	x_2	x_3	x_4	x_5
$-\xi$	0	0	0	0	1	1
x_1	2	1	0	$-2/7$	$1/14$	$2/7$
x_2	0	0	1	$8/7$	$3/14$	$-1/7$

We can start the second phase by removing the artificial variables and using the original objective function.

		x_1	x_2	x_3
$-z$	0	1	-1	-1
x_1	2	1	0	$-2/7$
x_2	0	0	1	$8/7$

We have to exclude the basic variables from the top row. This gives.

		x_1	x_2	x_3
$-z$	-2	0	0	$3/7$
x_1	2	1	0	$-2/7$
x_2	0	0	1	$8/7$

Since $a_{0,1}$, $a_{0,2}$ and $a_{0,3}$ are all non-negative this $(2, 0, 0)^T$ is an optimal basic feasible solution to the original problem.