Theorem 1. *Let*

$$\mathbf{T} = \left(\begin{array}{c|c} f & \mathbf{c}^T \\ \hline \\ \mathbf{b} & \mathbf{A} \end{array}\right)$$

be a tableau, and let $B = (j_1, \ldots, j_m)$ be an ordered basis. Let $\mathbf{X}$ be a non-singular matrix with first column $[1, 0, \ldots, 0]^T$. If

$$\mathbf{X}\left[\begin{array}{c} \mathbf{c}_B^T \\ \hline \mathbf{A}_B \end{array}\right] = \left[\begin{array}{c} \mathbf{0}^T \\ \hline \\ \mathbf{I} \end{array}\right],$$

then $\mathbf{X} = \left(\begin{array}{c|c} 1 & -\pi^T \\ \hline \\ \mathbf{0} & \mathbf{A}_B^{-1} \end{array}\right)$, *where $\pi^T = \mathbf{c}_B^T \mathbf{A}_B^{-1}$. In particular,*

$$\mathbf{XT} = \left[\begin{array}{c|c} f - \pi^T \mathbf{b} & \overline{\mathbf{c}}^T \\ \hline \\ \mathbf{A}_B^{-1}\mathbf{b} & \mathbf{A}_B^{-1}\mathbf{A} \end{array}\right].$$

where $\overline{\mathbf{c}}^T = \mathbf{c}^T - \pi^T \mathbf{A}$.

Remark. In the simplex procedure there are two row operations: either we multiple some row $i \in \{1, \ldots, n\}$ by α, or we add some multiple of a row $i \in \{1, \ldots, n\}$ to a row $j \in \{0, \ldots, n\}$. The first operation corresponds to pre-multiplication of $\mathbf{T}$ by $\mathbf{I}(i, \alpha)$ which is obtained from the identity matrix $\mathbf{I}$ by replacing 1 in the entry (i, i) with α, and the second operation corresponds to pre-multiplication of $\mathbf{T}$ by $\mathbf{I}(i, j, \alpha)$ which is obtained from the identity matrix $\mathbf{I}$ by replacing 0 in the entry (i, j) with α. This means that $\widetilde{\mathbf{T}}$ is obtained from $\mathbf{T}$ by a sequence of pre-multiplications by such matrices. Note that in all cases the matrix we pre-multiply by has the first column $(1, 0, 0, \ldots, 0)^T$, i.e. we never add a multiple of row 0 to row i where $i \in \{1, \ldots, n\}$ and we never multiply row 0 by a constant. Therefore, the product of these matrices has first column $(1, 0, 0, \ldots, 0)^T$ since the product of two matrices with the zero column $(1, 0, 0, \ldots, 0)^T$ also has the zero column $(1, 0, 0, \ldots, 0)^T$.

Proof. Let $\mathbf{X} = \left[\begin{array}{c|c} 1 & \mathbf{u}^T \\ \hline \mathbf{0} & \mathbf{U} \end{array}\right]$ so $\mathbf{X}\left[\begin{array}{c} \mathbf{c}_B \\ \hline \mathbf{A}_B \end{array}\right] = \left[\begin{array}{c} \mathbf{0}^T \\ \hline \\ \mathbf{I} \end{array}\right]$ gives the equations $\mathbf{c}_B^T + \mathbf{u}^T \mathbf{A}_B = \mathbf{0}^T$ and $\mathbf{U}\mathbf{A}_B = \mathbf{I}$. We then get $\mathbf{u}^T = -\mathbf{c}_B^T \mathbf{A}_B^{-1}$ and $\mathbf{U} = \mathbf{A}_B^{-1}$, by solving the matrix equations. This proves the first part. The "In particular" statement, follows from block multiplication. ☐