Simplex first example

Suppose we are given the problem

$$\text{Minimize } z = -x_1 + 2x_2 - x_3$$

subject to

$$\begin{cases} x_1 & -2x_2 & +x_3 & +x_4 & & = & 10 \\ 2x_1 & -3x_2 & -x_3 & & +x_5 & = & 6 \\ x_1, & x_2, & x_3, & x_4, & x_5 & \geq & 0. \end{cases}$$

Rewrite the objective function as a new equation with the new variable z and switch the LHS with the RHS.

$$\begin{cases} 0 & = & -z & -x_1 & +2x_2 & -x_3 & & & & \\ 10 & = & & x_1 & -2x_2 & +x_3 & +x_4 & & & \\ 6 & = & & 2x_1 & -3x_2 & -x_3 & & +x_5 & & \\ & & & x_1, & x_2, & x_3, & x_4, & x_5 & \geq & 0. \end{cases}$$

$$\begin{cases} 0 & = & -z & -x_1 & +2x_2 & -x_3 & & & & \\ 10 & = & & x_1 & -2x_2 & +x_3 & +x_4 & & & \\ 6 & = & & 2x_1 & -3x_2 & -x_3 & & +x_5 & & \\ & & & x_1, & x_2, & x_3, & x_4, & x_5 & \geq & 0. \end{cases}$$

Note this system is solved for the variables x_4 and x_5.
Set $x_0 = -z$.
We have the following initial tableau:
(Note: in the future we will not include the x_0 column in the tableau, so we refer to the entries in the x_1 column as $a_{0,1}, \dots, a_{m,1}$, the x_2 column as $a_{0,1}, \dots, a_{m,2}$, etc.):

		x_0	x_1	x_2	x_3	x_4	x_5
$x_0 = -z$	0	1	-1	2	-1	0	0
x_4	10	0	1	-2	1	1	0
x_5	6	0	2	-3	-1	0	1

The tableau is solved for the ordered basis $B = (4,5)$ and basic feasible solution $x = (0,0,0,10,6)^T$.

		x_0	$\mathbf{x_1}$	x_2	x_3	x_4	x_5
$x_0 = -z$	0	1	$-\mathbf{1}$	2	-1	0	0
x_4	10	0	$\mathbf{1}$	-2	1	1	0
x_5	6	0	$\mathbf{2}$	-3	-1	0	1

1. Check whether there are negative entries in the top row other than the 0th entry.
2. Choose ANY such negative entry, for example x_1. It's corresponding column is called the *pivot column*.
3. Both entries in this column are positive, so we must compare ratios to determine the pivot column.
4. The ratio $a_{1,0}/a_{1,1} = 10/1$ is greater than the ratio $a_{2,0}/a_{2,1} = 6/2$, so the *pivot row* will be Row 2. The entry in the pivot row (row 2) and pivot column (column 1) is called the *pivot entry*.
5. Pivoting on this pivot entry will solve the tableau for the basis where 5 is replaced with 1, so the new basis will be $(4,1)$.

		x_0	x_1	x_2	x_3	x_4	x_5
$x_0 = -z$	3	1	0	1/2	−3/2	0	1/2
x_4	7	0	0	−1/2	**3/2**	1	−1/2
x_1	3	0	1	−3/2	−1/2	0	1/2

1. This tableau corresponds to basis $B = (4, 1)$ and bfs $x = (3, 0, 0, 7, 0)^T$.
2. The tableau implies that $z = -3 + 1/2 \cdot x_2 - 3/2 \cdot x_3 + 1/2 \cdot x_5$, so $z = -3$ for the current bfs, and this solution is better than the previous one.
3. The only negative entry in the 0th row is the coefficient at x_3, so, the pivot column is the x_3 column.
4. In this column, there is only one positive entry, so we pivot on this entry, which will replace 4 with 3 in the basis.

		x_0	x_1	x_2	x_3	x_4	x_5
$x_0 = -z$	10	1	0	0	0	1	0
x_3	14/3	0	0	−1/3	1	2/3	−1/3
x_1	16/3	0	1	−5/3	0	1/3	1/3

1. This tableau corresponds to the basis $(3, 1)$, bfs $x = (16/3, 0, 14/3, 0, 0)^T$, and implies $z = -10 + x_4$.
2. Note that $z = -10 + x_4$ implies that $z \geq -10$, because for any feasible solution $x_4 \geq 0$, so the current bfs is an optimum and the algorithm terminates.
3. In general, the simplex algorithm terminates when all of the entries, except possibly the 0th entry, in the top row are non-negative.