Suppose we are given the problem

$$\text{Minimize } z = 2x_1 + 3x_2 + 4x_3 + 5x_4$$

subject to

$$\begin{cases} x_1 & -x_2 & +x_3 & -x_4 & \geq & 10, \\ x_1 & -2x_2 & +3x_3 & -4x_4 & \geq & 6, \\ 3x_1 & -4x_2 & +5x_3 & -6x_4 & \geq & 15 \\ x_1, & x_2, & x_3, & x_4 & \geq & 0. \end{cases} \tag{1}$$

- Do we add slack variables or surplus variables?
- Do we automatically have a basic feasible solution after adding surplus variables?
- We could do two phase simplex,
- but since the coefficient in the objective function are positive, we can use dual simplex.

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-z$	0	2	3	4	5	0	0	0
x_5	-10	-1	1	-1	1	1	0	0
x_6	-6	-1	2	-3	4	0	1	0
x_7	-15	-3	4	-5	6	0	0	1

- After adding surplus variables and multiplying the equations by -1 we have the that rows 1, 2 and 3 are solved for x_5, x_6 and x_7, respectively.
- The ordered basis is $(5, 6, 7)$, but it is NOT a feasible basis since $A_B^{-1}b = [-10, -6, -15]^T$ has negative entries. It is a dual feasible basis, because $\overline{c}^T \geq 0^T$, so we can use the dual simplex algorithm.
- In dual simplex, we pick the pivot row first by selecting a row with a negative entry in column 0.
- Therefore, any row in this tableau is acceptable. We pick Row 1.
- We now pick the pivot column so that $a_{0,0}$ does not increase and the top row remains positive. This means that if r is the pivot row, we select column s so that $a_{r,s} < 0$ and, subject to this, $a_{0,s}/a_{r,s}$ is as large as possible.
- Since the ratio for column x_1 is $a_{0,1}/a_{1,1} = \frac{2}{-1} = -2$ and the ratio for column x_3 is $a_{0,3}/a_{1,3} = \frac{4}{-1} = -4$, we pivot on $a_{1,1}$.

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-z$	-20	0	5	2	7	2	0	0
x_1	10	1	-1	1	-1	-1	0	0
x_6	4	0	1	-2	3	-1	1	0
x_7	15	0	1	-2	3	-3	0	1

- Now every $a_{i,0}$ for $i \in [m]$ is nonnegative. So, the tableau is optimal.
- But suppose that the boss adds a new restriction:

$$x_1 + 2x_2 + 3x_3 - 4x_4 \leq 8.$$

- With the dual simplex, we do not need to start from scratch. We simply add the new row and one more column to our final tableau.

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
$-z$	-20	0	5	2	7	2	0	0	0
x_1	10	1	-1	1	-1	-1	0	0	0
x_6	4	0	1	-2	3	-1	1	0	0
x_7	15	0	1	-2	3	-3	0	1	0
x_8	8	1	2	3	-4	0	0	0	1

- This tableau has a new row for the new equation and also a new slack variable
- We have to make sure that this tableau is solved for x_1, x_6 and x_7, so we must exclude them from the row we just added.

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
$-z$	-20	0	5	2	7	2	0	0	0
x_1	10	1	-1	1	-1	-1	0	0	0
x_6	4	0	1	-2	3	-1	1	0	0
x_7	15	0	1	-2	3	-3	0	1	0
x_8	-2	0	3	2	-3	1	0	0	1

- Notice how if in the last row we did not have -3, then the LP would be infeasible, because the left hand size of the equation would be less than zero and the right hand size would always be at least zero.
- Since there $a_{4,0} = -2$ and $a_{i,0} \geq 0$ for all $i \in [3]$, we must pivot on row 4. We are then forced to pivot on column x_4 because $a_{4,4} = 3$ is the only negative entry in row 4.

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
$-z$	$-74/3$	0	12	$20/3$	0	$13/3$	0	0	$7/3$
x_1	$32/3$	1	-2	$1/3$	0	$-4/3$	0	0	$-1/3$
x_6	2	0	4	0	0	0	1	0	1
x_7	13	0	4	0	0	-2	0	1	1
x_4	$2/3$	0	-1	$-2/3$	1	$-1/3$	0	0	$-1/3$

- Are we now done?
- Why is the optimal value of this LP (74/3) higher than the optimal value of the previous LP (20)?