Cycling example (2.7 from book)

- This is an example 2.7 from the book and is an example of cycling.
- The first tableau T_1 will appear again as tableau T_7 when we use the following natural pivot rules.
- Select the pivot column s so that $a_{0,s} = \overline{c}_s \leq \overline{c}_j = a_{0,j}$ for all $j \in [n]$ (In this example, this always gives a unique choice)
- In the case of ties when selecting the pivot row, select the row so that the smallest index leaves the basis (this rule is the same as Bland's rule)

Tableau 1

$$T_1 = \begin{array}{c|c|ccccccc}
 & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\
\hline
-z & 3 & -\frac{3}{4} & 20 & -\frac{1}{2} & 6 & 0 & 0 & 0 \\
\hline
x_5 & 0 & \mathbf{1/4} & -8 & -1 & 9 & 1 & 0 & 0 \\
x_6 & 0 & \frac{1}{2} & -12 & -\frac{1}{2} & 3 & 0 & 1 & 0 \\
x_7 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}$$

bfs $x = (0, 0, 0, 0, 0, 1)$, basis $B = (5, 6, 7)$.

Tableau 2

$$T_1 = \begin{array}{c|c|ccccccc}
 & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\
\hline
-z & 3 & -\frac{3}{4} & 20 & -\frac{1}{2} & 6 & 0 & 0 & 0 \\
\hline
x_5 & 0 & \frac{1}{4} & -8 & -1 & 9 & 1 & 0 & 0 \\
x_6 & 0 & \frac{1}{2} & -12 & -\frac{1}{2} & 3 & 0 & 1 & 0 \\
x_7 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}$$

bfs $x = (0, 0, 0, 0, 0, 1)$, basis $B = (5, 6, 7)$.

$$T_2 = \begin{array}{c|c|ccccccc}
 & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\
\hline
-z & 3 & 0 & -4 & -\frac{7}{2} & 33 & 3 & 0 & 0 \\
\hline
x_1 & 0 & 1 & -32 & -4 & 36 & 4 & 0 & 0 \\
x_6 & 0 & 0 & \mathbf{4} & \frac{3}{2} & -15 & -2 & 1 & 0 \\
x_7 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}$$

bfs $x = (0, 0, 0, 0, 0, 1)$, basis $B = (1, 6, 7)$.

Tableau 3

$T_1 =$

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-z$	3	$-\frac{3}{4}$	20	$-\frac{1}{2}$	6	0	0	0
x_5	0	$\frac{1}{4}$	-8	-1	9	1	0	0
x_6	0	$\frac{1}{2}$	-12	$-\frac{1}{2}$	3	0	1	0
x_7	1	0	0	1	0	0	0	1

bfs $x = (0,0,0,0,0,1)$, basis $B = (5,6,7)$.

$T_3 =$

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-z$	3	0	0	-2	18	1	1	0
x_1	0	1	0	$\mathbf{8}$	-84	-12	8	0
x_2	0	0	1	$\frac{3}{8}$	$-\frac{15}{4}$	$-\frac{1}{2}$	$\frac{1}{4}$	0
x_7	1	0	0	1	0	0	0	1

bfs $x = (0,0,0,0,0,1)$, basis $B = (1,2,7)$.

Tableau 4

$T_1 =$

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-z$	3	$-\frac{3}{4}$	20	$-\frac{1}{2}$	6	0	0	0
x_5	0	$\frac{1}{4}$	-8	-1	9	1	0	0
x_6	0	$\frac{1}{2}$	-12	$-\frac{1}{2}$	3	0	1	0
x_7	1	0	0	1	0	0	0	1

bfs $x = (0,0,0,0,0,1)$, basis $B = (5,6,7)$.

$T_4 =$

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-z$	3	$\frac{1}{4}$	0	0	-3	-2	3	0
x_3	0	$\frac{1}{8}$	0	1	$-\frac{21}{2}$	$-\frac{3}{2}$	1	0
x_2	0	$-\frac{3}{64}$	1	0	$\mathbf{\frac{3}{16}}$	$\frac{1}{16}$	$-\frac{1}{8}$	0
x_7	1	$-\frac{1}{8}$	0	0	$21/2$	$\frac{3}{2}$	-1	1

bfs $x = (0,0,0,0,0,1)$, basis $B = (3,2,7)$.

Tableau 5

$T_1 =$

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-z$	3	$-\frac{3}{4}$	20	$-\frac{1}{2}$	6	0	0	0
x_5	0	$\frac{1}{4}$	-8	-1	9	1	0	0
x_6	0	$\frac{1}{2}$	-12	$-\frac{1}{2}$	3	0	1	0
x_7	1	0	0	1	0	0	0	1

bfs $x = (0,0,0,0,0,1)$, basis $B = (5,6,7)$.

$T_5 =$

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-z$	3	$-\frac{1}{2}$	16	0	0	-1	1	0
x_3	0	$-\frac{5}{2}$	56	1	0	$\mathbf{2}$	-6	0
x_4	0	$-\frac{1}{4}$	$\frac{16}{3}$	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	0
x_7	1	$\frac{5}{2}$	-56	0	0	-2	6	1

bfs $x = (0,0,0,0,0,1)$, basis $B = (3,4,7)$.

Tableau 6

$$T_1 = \begin{array}{c|c|ccccccc} & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ \hline -z & 3 & -\frac{3}{4} & 20 & -\frac{1}{2} & 6 & 0 & 0 & 0 \\ \hline x_5 & 0 & \frac{1}{4} & -8 & -1 & 9 & 1 & 0 & 0 \\ x_6 & 0 & \frac{1}{2} & -12 & -\frac{1}{2} & 3 & 0 & 1 & 0 \\ x_7 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}$$

bfs $x = (0, 0, 0, 0, 0, 1)$, basis $B = (5, 6, 7)$.

$$T_6 = \begin{array}{c|c|ccccccc} & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ \hline -z & 3 & -\frac{7}{4} & 44 & \frac{1}{2} & 0 & 0 & -2 & 0 \\ \hline x_5 & 0 & -\frac{5}{4} & 28 & \frac{1}{2} & 0 & 1 & -3 & 0 \\ x_4 & 0 & \frac{1}{6} & -4 & -\frac{1}{6} & 1 & 0 & \frac{1}{3} & 0 \\ x_7 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}$$

bfs $x = (0, 0, 0, 0, 0, 1)$, basis $B = (5, 4, 7)$.

Tableau 7 same as Tableau 1!

$$T_1 = \begin{array}{c|c|ccccccc} & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ \hline -z & 3 & -\frac{3}{4} & 20 & -\frac{1}{2} & 6 & 0 & 0 & 0 \\ \hline x_5 & 0 & \frac{1}{4} & -8 & -1 & 9 & 1 & 0 & 0 \\ x_6 & 0 & \frac{1}{2} & -12 & -\frac{1}{2} & 3 & 0 & 1 & 0 \\ x_7 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}$$

bfs $x = (0, 0, 0, 0, 0, 1)$, basis $B = (5, 6, 7)$.

$$T_7 = \begin{array}{c|c|ccccccc} & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ \hline -z & 3 & -\frac{3}{4} & 20 & -\frac{1}{2} & 6 & 0 & 0 & 0 \\ \hline x_5 & 0 & \frac{1}{4} & -8 & -1 & 9 & 1 & 0 & 0 \\ x_6 & 0 & \frac{1}{2} & -12 & -\frac{1}{2} & 3 & 0 & 1 & 0 \\ x_7 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}$$

bfs $x = (0, 0, 0, 0, 0, 1)$, basis $B = (5, 6, 7)$.

Example with Lexicographic simplex

- We start with the same tableau as example 2.7 from the book, but this time we following the lexicographic simplex method.
- That is, in the case of ties when selecting the pivot row, we select the row that is smallest in lexicographic order.

Tableau 1

$$
T_1 = \begin{array}{c|c|ccccccc}
 & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\
\hline
-z & 3 & -\frac{3}{4} & 20 & -\frac{1}{2} & 6 & 0 & 0 & 0 \\
\hline
x_5 & 0 & \frac{1}{4} & -8 & -1 & 9 & 1 & 0 & 0 \\
x_6 & 0 & \frac{1}{2} & -12 & -\frac{1}{2} & 3 & 0 & 1 & 0 \\
x_7 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}
$$

bfs $x = (0, 0, 0, 0, 0, 1)^T$, basis $B = (5, 6, 7)$.

- We pivot on column 1.
- Rows 1 and 2 are tied, by the standard simplex pivot row selection rule.
- But row 1 divided by $a_{1,1}$ is lexicographically less that row 2 divied by $a_{2,1}$ ($a_{1,3}/a_{1,1} = -32 < -24 = a_{2,3}/a_{2,1}$),
- so we pick row 1 since we are doing lexicographic simplex.
- this choice is the same as the previous example.

Tableau 2

$$
T_2 = \begin{array}{c|c|ccccccc}
 & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\
\hline
-z & 3 & 0 & -4 & -\frac{7}{2} & 33 & 3 & 0 & 0 \\
\hline
x_1 & 0 & 1 & -32 & -4 & 36 & 4 & 0 & 0 \\
x_6 & 0 & 0 & \mathbf{4} & \frac{3}{2} & -15 & -2 & 1 & 0 \\
x_7 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}
$$

bfs $x = (0, 0, 0, 0, 0, 1)^T$, basis $B = (1, 6, 7)$.

- We pivot on column 2.
- There is no choice for the pivot row,
- so this step is also the same as the example in the book.

Tableau 3

$$
T_3 = \begin{array}{c|c|ccccccc}
 & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\
\hline
-z & 3 & 0 & 0 & -2 & 18 & 1 & 1 & 0 \\
\hline
x_1 & 0 & 1 & 0 & 8 & -84 & -12 & 8 & 0 \\
x_2 & 0 & 0 & 1 & \mathbf{\frac{3}{8}} & -\frac{15}{4} & -\frac{1}{2} & \frac{1}{4} & 0 \\
x_7 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}
$$

bfs $x = (0, 0, 0, 0, 0, 1)^T$, basis $B = (1, 2, 7)$.

- We select 3 as the pivot column, it is the only possible pivot column
- In this step, rows 1 and 2 are tied, and we select row 2 because it is lexicographically less than row 1, ($a_{1,0}/a_{1,3} = a_{2,0}/a_{2,3} = 0$, and $a_{2,1}/a_{2,3} = 0 < 1/8 = a_{1,1}/a_{1,3}$). this step is different than the example in the book.
- Note that if we were following Bland's rule, we would pick row one instead of row two. This is because row 1 is solved for x_1 and row 2 is solved for x_2 and x_1 has a lower index than x_2.

Tableau 4

$$T_4 = \begin{array}{r|c|ccccccc|} & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ \hline -z & 3 & 0 & 16/3 & 0 & -2 & -5/3 & 7/3 & 0 \\ \hline x_1 & 0 & 1 & -64/3 & 0 & -4 & -4/3 & 8/3 & 0 \\ x_3 & 0 & 0 & 8/3 & 1 & -10 & -4/3 & 2/3 & 0 \\ x_7 & 1 & 0 & -8/3 & 0 & 10 & \mathbf{4/3} & -2/3 & 1 \end{array}$$

bfs $x = (0, 0, 0, 0, 0, 1)^T$, basis $B = (3, 2, 7)$.

- We pick column 5 as our pivot column, but x_4 would also be a valid choice.
- Once we select column 5 as our pivot column, we must pivot on row 3

Tableau 5

$$T_5 = \begin{array}{r|c|ccccccc|} & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ \hline -z & 17/4 & 0 & 2 & 0 & 21/2 & 0 & 3/2 & 5/4 \\ \hline x_1 & 1 & 1 & -24 & 0 & 6 & 0 & 2 & 1 \\ x_3 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ x_5 & 3/4 & 0 & -2 & 0 & 15/2 & 1 & -1/2 & 3/4 \end{array}$$

bfs $x = (1, 0, 1, 0, 3/4, 0)^T$, basis $B = (1, 3, 5)$.

- This is the optimal solution, because the entries in the top row (columns 1 thru 7) are non-negative.

Quiz

- Find the pivot entry using Bland's rule and lexicographic simplex

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-z$	8	2	0	0	2	-3	0	0
x_7	2	2	0	0	3	4	0	1
x_6	6	10	0	0	4	12	1	0
x_3	4	5	0	1	2	8	0	0
x_2	2	0	1	0	4	3	0	0

- Using Bland's rule we select column x_5 and row 3
- Using lexicographic simplex we select column x_5 and row 1