Theorem 1. *Suppose we have the following LP in standard form*

$$\text{(P)} \qquad \min \mathbf{c}^T\mathbf{x} \;\; s.t \;\; \mathbf{A}\mathbf{x} = \mathbf{b} \;\; and \;\; \mathbf{x} \geq \mathbf{0},$$

and that $B = (j_1, \dots, j_m)$ is a basis.

(a) The vector $\mathbf{x}$ defined by $\mathbf{x}_B = \mathbf{A}_B^{-1}\mathbf{b}$ and $x_j = 0$ for every nonbasic variable x_j is the unique basic solution associated with B. In particular, B is a feasible basis when $\mathbf{A}_B^{-1}\mathbf{b} \geq \mathbf{0}$.
(b) If B is a feasible basis and the relative cost vector is non-negative (i.e. $\overline{\mathbf{c}}^T \geq \mathbf{0}^T$), then x is an optimum for (P)*.*
(c) If B is a feasible basis and there exists a variable x_{j_0} such that the relative cost of x_{j_0} is negative (i.e. $\overline{c}_{j_0} < 0$) and $\mathbf{A}_B^{-1}\mathbf{A}_{j_0} \leq \mathbf{0}$, then (P) *is unbounded.*

Proof. We begin by proving (a). By the definition, $\mathbf{x}$ is a basic solution associated with the basis B if and only if $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $x_j = 0$ for every nonbasic variable x_j. Using these two facts we have that

$$\begin{aligned}
\mathbf{b} &= \mathbf{A}\mathbf{x} \\
&= \mathbf{A}_1 x_1 + \cdots + \mathbf{A}_n x_n \\
&= \mathbf{A}_{j_1} x_{j_1} + \cdots + \mathbf{A}_{j_m} x_{j_m} \\
&= \left(\mathbf{A}_{j_1} \mid \cdots \mid \mathbf{A}_{j_m} \right) \begin{pmatrix} x_{j_1} \\ \vdots \\ x_{j_m} \end{pmatrix} \\
&= \mathbf{A}_B \mathbf{x}_B.
\end{aligned}$$

Furthermore, because B is a basis, $\mathbf{A}_B$ is an invertible matrix. This implies that $\mathbf{A}_B\mathbf{x}_B = \mathbf{b}$ if and only if $\mathbf{x}_B = \mathbf{A}_B^{-1}\mathbf{b}$. Therefore, $\mathbf{x}$ is a basic solution associated with the basis B if and only if $\mathbf{x}_B = \mathbf{A}_B^{-1}\mathbf{b}$ and $x_j = 0$ for every nonbasic variable x_j. This proves (a).

To prove (b), first note that, since $x_j = 0$ for every nonbasic variable x_j,

$$(*) \qquad \begin{aligned}
\mathbf{c}^T\mathbf{x} &= c_1 x_1 + \dots c_n x_n \\
&= c_{j_1} x_{j_1} + \dots c_{j_m} x_{j_m} \\
&= \left(c_{j_1}, \dots, c_{j_m} \right) \begin{pmatrix} x_{j_1} \\ \vdots \\ x_{j_m} \end{pmatrix} \\
&= \mathbf{c}_B^T \mathbf{x}_B.
\end{aligned}$$

Now let $\mathbf{x}'$ be an arbitrary feasible solution to (P). We will show that $\mathbf{c}^T\mathbf{x}' \geq \mathbf{c}^T\mathbf{x}$ which will prove (b). By the definition of a feasible solution, we have that $\mathbf{A}\mathbf{x}' = \mathbf{b}$ and $\mathbf{x}' \geq \mathbf{0}$. Using these two facts, the assumption that $\overline{\mathbf{c}}^T \geq \mathbf{0}^T$ and that fact that, by (a), $\mathbf{x}_B = \mathbf{A}_B^{-1}b$, we can compute that

$$\begin{aligned}0 &\leq \overline{\mathbf{c}}^T\mathbf{x}' \\ &= (\mathbf{c}^T - \mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{A})\mathbf{x}' \\ &= \mathbf{c}^T\mathbf{x}' - (\mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{A})\mathbf{x}' \\ &= \mathbf{c}^T\mathbf{x}' - \mathbf{c}_B^T\mathbf{A}_B^{-1}(\mathbf{A}\mathbf{x}') \\ &= \mathbf{c}^T\mathbf{x}' - \mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{b} \\ &= \mathbf{c}^T\mathbf{x}' - \mathbf{c}_B^T\mathbf{x}_B,\end{aligned}$$

This, with (*), gives us that

$$\mathbf{c}^T\mathbf{x}' \geq \mathbf{c}_B^T\mathbf{x}_B = \mathbf{c}^T\mathbf{x},$$

which proves (b).

Finally, we prove (c). First, recall that $\mathbf{x}$ is a feasible solution. We will construct $\mathbf{y}$ such that $\mathbf{A}\mathbf{y} = \mathbf{0}$, $\mathbf{y} \geq \mathbf{0}$ and $\mathbf{c}^T\mathbf{y} < 0$. This with question 4 on homework assignment 1, will prove (c). First note that x_{j_0} cannot be a basic variable, because for every $i \in [m]$, the relative cost of the basic variable x_{j_i} is

$$\overline{c}_{j_i} = c_{j_i} - \mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{A}_{j_i} = c_{j_i} - \mathbf{c}_B^T(\mathbf{A}_B^{-1}\mathbf{A}_{j_i}) = c_{j_i} - \begin{pmatrix} c_{j_1}, \ldots, c_{j_m} \end{pmatrix}\mathbf{e}_i = c_{j_i} - c_{j_i} = 0,$$

and $\overline{c}_{j_0} < 0$ by assumption. Here we used that fact that $\mathbf{A}_{j_i}$ is the ith column of $\mathbf{A}_B$, so the product $\mathbf{A}_B^{-1}\mathbf{A}_{j_i}$ must be the ith standard basis vector which we denote by $\mathbf{e}_i$. Therefore, we can set the variables $y_{j_1}, \ldots, y_{j_m}$ of $\mathbf{y}$ by $\mathbf{y}_B = -\mathbf{A}_B^{-1}\mathbf{A}_{j_0}$ and set $y_{j_0} = 1$ without conflict. We set the entries in $\mathbf{y}$ that we have not specified to 0, i.e. we set $y_j = 0$ for every nonbasic variable x_j that is not x_{j_0}. Note that $\mathbf{y} \geq \mathbf{0}$ because $\mathbf{A}_B^{-1}\mathbf{A}_{j_0} \leq \mathbf{0}$ by assumption. We now show that $\mathbf{y}$ is in the null space of $\mathbf{A}$ with the following computation

$$\mathbf{A}\mathbf{y} = \mathbf{A}_B\mathbf{y}_B - \mathbf{A}_{j_0}y_{j_0} = \mathbf{A}_B(\mathbf{A}_B^{-1}\mathbf{A}_{j_0}) - \mathbf{A}_{j_0} = (\mathbf{A}_B\mathbf{A}_B^{-1})\mathbf{A}_{j_0} - \mathbf{A}_{j_0} = \mathbf{A}_{j_0} - \mathbf{A}_{j_0} = \mathbf{0}.$$

Finally, because

$$\mathbf{c}^T\mathbf{y} = \mathbf{c}_B^T\mathbf{y}_B + c_{j_0}y_{j_0} = \mathbf{c}_B^T(-\mathbf{A}_B^{-1}\mathbf{A}_{j_0}) + c_{j_0} = c_{j_0} - \mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{A}_{j_0} = \overline{c}_{j_0}$$

we have that $\mathbf{c}^T\mathbf{y} = \overline{c}_{j_0} < 0$ which finishes the proof. $\square$