Simplex - first example

Suppose we are given the problem

$$\text{Minimize } z = -x_1 + 2x_2 - x_3$$

subject to

$$\begin{cases} x_1 & -2x_2 & +x_3 & +x_4 & & = & 10 \\ 2x_1 & -3x_2 & -x_3 & & +x_5 & = & 6 \\ x_1, & x_2, & x_3, & x_4, & x_5 & \geq & 0. \end{cases}$$

- Rewrite the objective function as $0 = -z - x_1 + 2x_2 - x_3$.
- For the constraints, swap the LHS and RHS.
- We have the following:

$$\begin{cases} 0 & = & -z & -x_1 & +2x_2 & -x_3 & & & \\ 10 & = & & x_1 & -2x_2 & +x_3 & +x_4 & & \\ 6 & = & & 2x_1 & -3x_2 & -x_3 & & +x_5 & \\ & & & x_1, & x_2, & x_3, & x_4, & x_5 & \geq 0. \end{cases}$$

$$\begin{cases} 0 & = & -z & -x_1 & +2x_2 & -x_3 & & & \\ 10 & = & & x_1 & -2x_2 & +x_3 & +x_4 & & \\ 6 & = & & 2x_1 & -3x_2 & -x_3 & & +x_5 & \\ & & & x_1, & x_2, & x_3, & x_4, & x_5 & \geq 0. \end{cases}$$

Set $x_0 = -z$. We have the following initial tableau:

		x_0	x_1	x_2	x_3	x_4	x_5
$x_0 = -z$	0	1	-1	2	-1	0	0
x_4	10	0	1	-2	1	1	0
x_5	6	0	2	-3	-1	0	1

The tableau is solved for the basis $B = (4, 5)$ and the corresponding basic feasible solution is $\mathbf{x} = (0, 0, 0, 10, 6)^T$. (Note: in the future we will not include the x_0 column in the tableau. We refer to the entries in the first column as, $\begin{bmatrix} a_{0,0} \\ \vdots \\ a_{m,0} \end{bmatrix}$, the x_1 column as $\begin{bmatrix} a_{0,1} \\ \vdots \\ a_{m,1} \end{bmatrix}$, etc.).

		x_0	$\mathbf{x_1}$	x_2	x_3	x_4	x_5
$x_0 = -z$	0	1	$\mathbf{-1}$	2	-1	0	0
x_4	10	0	$\mathbf{1}$	-2	1	1	0
x_5	6	0	$\mathbf{2}$	-3	-1	0	1

1. Check for negative entries in the top row other than $a_{0,0}$.
2. Choose ANY such negative entry, for example $a_{0,1} = -1$. The corresponding column is called the *pivot column*.
3. Both entries in this column are positive, so we must compare ratios to determine the pivot column.
4. The ratio $\frac{a_{1,0}}{a_{1,1}} = \frac{10}{1}$ is greater than the ratio $\frac{a_{2,0}}{a_{2,1}} = \frac{6}{2}$, so the *pivot row* is row 2. The entry $a_{2,1} = 2$ is the called the *pivot entry*.
5. Pivoting on the pivot entry will solve the tableau for the basis $(4, 1)$. We replace 5 with 1 in the basis, because the x_1 column is the pivot column and the pivot row is solved for x_5.

		x_0	x_1	x_2	x_3	x_4	x_5
$x_0 = -z$	3	1	0	$1/2$	$-3/2$	0	$1/2$
x_4	7	0	0	$-1/2$	$\mathbf{3/2}$	1	$-1/2$
x_1	3	0	1	$-3/2$	$-1/2$	0	$1/2$

1. This tableau is solved for the basis $B = (4, 1)$ and corresponding bfs is $\mathbf{x} = (3, 0, 0, 7, 0)^T$.
2. The tableau implies that $z = -3 + \frac{1}{2} \cdot x_2 - \frac{3}{2} \cdot x_3 + \frac{1}{2} \cdot x_5$, so $z = -3$ for the current bfs, and this solution is better than the previous one.
3. The only negative entry in the 0th row is the coefficient at x_3, $a_{0,3} = -3/2$, so the pivot column is the x_3 column.
4. In this column, there is only one positive entry, $a_{1,3} = 3/2$, so we pivot on this entry to solve for the basis $(3, 1)$.

		x_0	x_1	x_2	x_3	x_4	x_5
$x_0 = -z$	10	1	0	0	0	1	0
x_3	$14/3$	0	0	$-1/3$	1	$2/3$	$-1/3$
x_1	$16/3$	0	1	$-5/3$	0	$1/3$	$1/3$

1. This tableau corresponds to the basis $(3, 1)$ and the correspond bfs is $\mathbf{x} = (16/3, 0, 14/3, 0, 0)^T$.
2. We have that $z = -10 + x_4$ which implies that $z \geq -10$, because in any feasible solution $x_4 \geq 0$. Because the value of the object function at the current bfs is -10, the current bfs is an optimum and the algorithm terminates.
3. In general, *the simplex algorithm terminates when all of the entries in the top row, except possibly $a_{0,0}$, are non-negative.* (Note: If the LP is unbounded there is a different condition which causes the algorithm to terminate).

1. Recall our initial tableau $\mathbf{T}$ - now written without the headings

$$\mathbf{T} = \left(\begin{array}{r|r|rrrrr} 0 & 1 & -1 & 2 & -1 & 0 & 0 \\ \hline 10 & 0 & 1 & -2 & 1 & 1 & 0 \\ 6 & 0 & 2 & -3 & -1 & 0 & 1 \end{array}\right)$$

2. In the first pivot step, we multiplied row 2 by $1/2$ and then added row 2 to row 0 and then subtracted row 2 from row 1.
3. These three steps can be thought for a left multiplication or pre-multiplication by the following three invertible matrices.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

4. Similarly, the second pivot step is equivalent to pre-multiplication by the following three invertible matrices.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{3}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & \frac{1}{3} & 1 \end{pmatrix}$$

1. So the final tableau $\mathbf{T}'$ is the initial tableau $\mathbf{T}$ pre-multiplied by the following invertible matrix $\mathbf{X}$

$$\mathbf{X} := \begin{pmatrix} 1 & 1 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & \frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

2. That is,

$$\mathbf{XT} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \left(\begin{array}{r|c|rrrrr} 0 & 1 & -1 & 2 & -1 & 0 & 0 \\ \hline 10 & 0 & 1 & -2 & 1 & 1 & 0 \\ 6 & 0 & 2 & -3 & -1 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{c|c|cccccc} 10 & 1 & 0 & 0 & 0 & 1 & 0 \\ \hline \frac{14}{3} & 0 & 0 & -\frac{1}{3} & 1 & \frac{2}{3} & -\frac{1}{3} \\ \frac{16}{3} & 0 & 1 & -\frac{5}{3} & 0 & \frac{1}{3} & \frac{1}{3} \end{array}\right) = \mathbf{T}'$$

3. Question: Why is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \mathbf{e}_1$ the first column of $\mathbf{X}$?

   Because the x_0 column of every tableau must be $\mathbf{e}_1$.

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1. From now on, let's remove the x_0 column from $\mathbf{T}$ since it never changes, so

$$\mathbf{T} = \left(\begin{array}{r|rrrrr} 0 & -1 & 2 & -1 & 0 & 0 \\ \hline 10 & 1 & -2 & 1 & 1 & 0 \\ 6 & 2 & -3 & -1 & 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} f & \mathbf{c}^T \\ \hline \mathbf{b} & \mathbf{A} \end{array}\right)$$

2. Using block multiplication to compute $\mathbf{T}' = \mathbf{XT}$ we get

$$\mathbf{XT} = \left(\begin{array}{c|cc} 1 & 1 & 0 \\ \hline 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} \end{array}\right) \left(\begin{array}{c|c} f & \mathbf{c}^T \\ \hline \mathbf{b} & \mathbf{A} \end{array}\right)$$

$$= \left(\begin{array}{c|c} (1)(f) + \begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{b} & (1)\mathbf{c}^T + \begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{A} \\ \hline \begin{pmatrix} 0 \\ 0 \end{pmatrix}(f) + \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}\mathbf{b} & \begin{pmatrix} 0 \\ 0 \end{pmatrix}\mathbf{c}^T + \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}\mathbf{A} \end{array}\right)$$

$$= \left(\begin{array}{c|c} f + \begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{b} & \mathbf{c}^T + \begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{A} \\ \hline \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}\mathbf{b} & \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}\mathbf{A} \end{array}\right)$$

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1. In general, we can write $\mathbf{X}$ as $\left(\begin{array}{c|c} 1 & \mathbf{d}^T \\ \hline 0 & \\ \vdots & \mathbf{U} \\ 0 & \end{array}\right) = \left(\begin{array}{c|c} 1 & \mathbf{d}^T \\ \hline \mathbf{0} & \mathbf{U} \end{array}\right)$.
2. So,

$$\mathbf{XT} = \left(\begin{array}{c|c} 1 & \mathbf{d}^T \\ \hline \mathbf{0} & \mathbf{U} \end{array}\right) \left(\begin{array}{c|c} f & \mathbf{c}^T \\ \hline \mathbf{b} & \mathbf{A} \end{array}\right) = \left(\begin{array}{c|c} f + \mathbf{d}^T\mathbf{b} & \mathbf{c}^T + \mathbf{d}^T\mathbf{A} \\ \hline \mathbf{Ub} & \mathbf{UA} \end{array}\right)$$

$$= \left(\begin{array}{c|c|c|c|c} f + \mathbf{d}^T\mathbf{b} & c_1 + \mathbf{d}^T\mathbf{A}_1 & c_2 + \mathbf{d}^T\mathbf{A}_2 & \dots & c_n + \mathbf{d}^T\mathbf{A}_n \\ \hline \mathbf{Ub} & \mathbf{UA}_1 & \mathbf{UA}_2 & \dots & \mathbf{UA}_n \end{array}\right)$$

3. Suppose $\mathbf{T}' = \mathbf{XT}$ is solved for the basis $B = (j_1, \dots, j_m)$.
4. Question: What is $\mathbf{UA}_B = \left(\begin{array}{c|c|c|c} \mathbf{UA}_{j_1} & \mathbf{UA}_{j_2} & \dots & \mathbf{UA}_{j_m} \end{array}\right)$?
   $\mathbf{UA}_B = \mathbf{I}_m$, because the last tableau is solved for B.
5. Question: What does that imply about $\mathbf{U}$? $\mathbf{U} = \mathbf{A}_B^{-1}$
6. Question: What is
   $\mathbf{c}_B^T + \mathbf{d}^T\mathbf{A}_B = \left(\begin{array}{c|c|c} c_{j_1} + \mathbf{d}^T\mathbf{A}_{j_1} & \dots & c_{j_m} + \mathbf{d}^T\mathbf{A}_{j_m} \end{array}\right)$?
   $\mathbf{c}_B^T + \mathbf{d}^T\mathbf{A}_B = \begin{pmatrix} 0 & 0 & \dots & 0 \end{pmatrix} = \mathbf{0}^T$.
7. Question: What does that imply about $\mathbf{d}^T$? $\mathbf{d}^T = -\mathbf{c}_B^T\mathbf{A}_B^{-1}$

You should know this!

1. To solve a tableau $\mathbf{T} = \left(\begin{array}{c|c} f & \mathbf{c}^T \\ \hline \mathbf{b} & \mathbf{A} \end{array}\right)$ for the basis B, pre-multiply by
$$\mathbf{X} = \left(\begin{array}{c|c} 1 & -\mathbf{c_B}^T \mathbf{A}_B^{-1} \\ \hline \mathbf{0} & \mathbf{A}_B^{-1} \end{array}\right) = \left(\begin{array}{c|c} 1 & \mathbf{c_B}^T \\ \hline \mathbf{0} & \mathbf{A}_B \end{array}\right)^{-1}.$$
2. The tableau solved for the basis B can be written as
$$\mathbf{XT} = \left(\begin{array}{c|c} f - \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{b} & \mathbf{c}^T - \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{A} \\ \hline \mathbf{A}_B^{-1} \mathbf{b} & \mathbf{A}_B^{-1} \mathbf{A} \end{array}\right).$$
3. We often define $\pi^T := \mathbf{c}_B^T \mathbf{A}_B^{-1}$, so $\mathbf{X} = \left(\begin{array}{c|c} 1 & -\pi^T \\ \hline \mathbf{0} & \mathbf{A}_B^{-1} \end{array}\right)$ and
$$\mathbf{XT} = \left(\begin{array}{c|c} f - \pi^T \mathbf{b} & \mathbf{c}^T - \pi^T \mathbf{A} \\ \hline \mathbf{A}_B^{-1} \mathbf{b} & \mathbf{A}_B^{-1} \mathbf{A} \end{array}\right).$$
4. Furthermore, we let $\bar{\mathbf{c}}^T = \mathbf{c}^T - \pi^T \mathbf{A}$. We call $\bar{\mathbf{c}}^T$ the *relative cost vector* and $\bar{c}_j$ the *relative cost* of x_j or the *relative cost* of Column j.