Suppose we are given the problem

$$\text{Minimize } z = -19x_1 - 13x_2 - 12x_3 - 17x_4$$

subject to

$$\begin{cases} 3x_1 & +2x_2 & +x_3 & +2x_4 & = & 225, \\ x_1 & +x_2 & +x_3 & +x_4 & = & 117, \\ 4x_1 & +3x_2 & +3x_3 & +4x_4 & = & 420 \\ x_1, & x_2, & x_3, & x_4 & \geq & 0. \end{cases} \tag{1}$$

- There is no obvious bfs, so we use the revised two phase simplex method
- To start the first phase, we add to each of the equations its own variable y_i and consider the auxiliary problem of minimizing $\xi = y_1 + y_2 + y_3$ (we think of $y_1 = x_5$, $y_2 = x_6$ and $y_3 = x_7$)
- Throughout the first phase, $\mathbf{c}^T$ and $\mathbf{A}$ refer to the cost vector and matrix of the first phase linear program, not the original LP (1).
- In the second phase, $\mathbf{c}^T$ and $\mathbf{A}$ refer to the cost vector and matrix of the original LP (1).

- This is the tableau corresponding to the phase one LP

		x_1	x_2	x_3	x_4	$x_5 = y_1$	$x_6 = y_2$	$x_7 = y_3$
$-\xi$	0	0	0	0	0	1	1	1
y_1	225	3	2	1	2	1	0	0
y_2	117	1	1	1	1	0	1	0
y_3	420	4	3	3	4	0	0	1

- Since $\mathbf{b} \geq \mathbf{0}$, we can use $B = (5, 6, 7)$ as our basis.
- We **do not** exclude y_1,y_2 and y_3 from the top row.
- Our carry matrix always has the form $\left[\begin{array}{c|c} -\pi^T\mathbf{b} & -\pi^T \\ \hline \mathbf{A}_B^{-1}\mathbf{b} & \mathbf{A}_B^{-1} \end{array}\right]$.
- When $B = (5, 6, 7)$, $\mathbf{A}_B = \mathbf{I}_m$, so $\mathbf{A}_B^{-1} = \mathbf{I}_m$, $\pi^T = \mathbf{c}_B^T\mathbf{A}_B^{-1} = \mathbf{c}_B^T = [1, 1, 1]$, $\mathbf{A}_B^{-1}\mathbf{b} = \mathbf{b} = [225, 117, 420]^T$, and $\pi^T\mathbf{b} = [1, 1, 1][255, 117, 420]^T = 225 + 117 + 420 = 762$.
- The following is our initial carry matrix

CARRY-0

$-\xi$	-762	-1	-1	-1
y_1	225	1	0	0
y_2	117	0	1	0
y_3	420	0	0	1

CARRY-0

$-\xi$	-762	-1	-1	-1		-8
y_1	225	1	0	0		3
y_2	117	0	1	0		1
y_3	420	0	0	1		4

- $\bar{c}_1 = c_1 - \pi^T\mathbf{A}_1 = 0 + [-1, -1, -1][3, 1, 4]^T = -8 < 0$, so we pivot on column 1.
- $\mathbf{A}_B^{-1}\mathbf{A}_1 = \mathbf{A}_1 = [3, 1, 4]^T$, so we append column $[-8, 3, 1, 4]^T$ to CARRY-0 and pivot on row one because $\frac{225}{3} < \frac{117}{1}$ and $\frac{225}{3} < \frac{420}{4}$.
- This means that, in the CARRY-0 matrix only, we divide row one by 3, then we add 8 times row one to row zero, add -1 times row one to row two, and add -4 times row one to row three.
- This yields the next carry matrix,

CARRY-1

		y_1	y_2	y_3
$-\xi$	-162	$5/3$	-1	-1
x_1	75	$1/3$	0	0
y_2	42	$-1/3$	1	0
y_3	120	$-4/3$	0	1

CARRY-1	$-\xi$	-162	$5/3$	-1	-1	$-2/3$
	x_1	75	$1/3$	0	0	$2/3$
	y_2	42	$-1/3$	1	0	$1/3$
	y_3	120	$-4/3$	0	1	$1/3$

- Now we calculate $\bar{c}_2 = c_2 - \pi^T \mathbf{A}_2 = 0 + [5/3, -1, -1][2, 1, 3]^T = -2/3$ (we do not calculate $\bar{c}_1$, because x_1 is basic which implies that $\bar{c}_1 = 0$.)
- Since $\bar{c}_2 = -2/3 < 0$, we pivot on column 2.
- We compute $\mathbf{A}_B^{-1}\mathbf{A}_2 = \begin{pmatrix} 1/3 & 0 & 0 \\ -1/3 & 1 & 0 \\ -4/3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/3 \\ 1/3 \end{pmatrix}$.
- Adding column $[-2/3, 2/3, 1/3, 1/3]^T$ to CARRY-1 and pivoting on the first row we get CARRY-2:

CARRY-2	$-\xi$	-87	2	-1	-1
	x_2	$225/2$	$1/2$	0	0
	y_2	$9/2$	$-1/2$	1	0
	y_3	$165/2$	$-3/2$	0	1

CARRY-2	$-\xi$	-87	2	-1	-1	-2
	x_2	$225/2$	$1/2$	0	0	$1/2$
	y_2	$9/2$	$-1/2$	1	0	$1/2$
	y_3	$165/2$	$-3/2$	0	1	$1/2$

- Since x_2 is in the basis and x_1 was removed from the basis on the previous step, we can start with column 3. (On the **next** iteration, we will **have to check** the x_1 column again.)
- We compute $\bar{c}_3 = c_3 - \pi^T \mathbf{A}_3 = 0 + [2, -1, -1][1, 1, 3]^T = -2 < 0$, so we pivot on column 3.
- Now we compute $\mathbf{A}_B^{-1}\mathbf{A}_3 = \begin{pmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ -3/2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 3/2 \end{pmatrix}$.
- Adding column $[-2, 1/2, 1/2, 3/2]^T$ to CARRY-2 and pivoting on the second row we get CARRY-3:

CARRY-3	$-\xi$	-69	0	3	-1
	x_2	108	1	-1	0
	x_3	9	-1	2	0
	y_3	69	0	-3	1

CARRY-3	$-\xi$	-69	0	3	-1	-1
	x_2	108	1	-1	0	2
	x_3	9	-1	2	0	-1
	y_3	69	0	-3	1	1

- We know must check column 1 again, so we compute $\bar{c}_1 = c_1 - \pi^T \mathbf{A}_1 = 0 + [0, 3, -1][3, 1, 4]^T = -1 < 0$, so we pivot on column 1.
- $\mathbf{A}_B^{-1}\mathbf{A}_1 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & -3 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$.
- We add column $[-1, 2, -1, 1]^T$ to CARRY-3 and pivot on the first row to get CARRY-4:

CARRY-4	$-\xi$	-15	$1/2$	$5/2$	-1
	x_1	54	$1/2$	$-1/2$	0
	x_3	63	$-1/2$	$3/2$	0
	y_3	15	$-1/2$	$-5/2$	1

CARRY-4	$-\xi$	-15	$1/2$	$5/2$	-1	$-1/2$
	x_1	54	$1/2$	$-1/2$	0	$1/2$
	x_3	63	$-1/2$	$3/2$	0	$1/2$
	y_3	15	$-1/2$	$-5/2$	1	$1/2$

- Note that x_1 entered the basis, then left it, and now entered it again.
- Since x_1 and x_3 are in the basis and x_2 was just removed from the basis on the last iteration, we can start with column 4: $\bar{c}_4 = c_4 - \pi^T \mathbf{A}_4 = 0 + [1/2, 5/2, -1][2, 1, 4]^T = -1/2 < 0$. So we pivot on column 4,
- $\mathbf{A}_B^{-1}\mathbf{A}_4 = \begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 3/2 & 0 \\ -1/2 & -5/2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$.
- Adding column $[-1/2, 1/2, 1/2, 1/2]^T$ to the last tableau and pivoting on the last row we get CARRY-5:

CARRY-5	$-\xi$	0	0	0	0
	x_1	39	1	2	-1
	x_3	48	0	4	-1
	x_4	30	-1	-5	2

CARRY-5	$-\xi$	0	0	0	0
	x_1	39	1	2	-1
	x_3	48	0	4	-1
	x_4	30	-1	-5	2

- Since $\xi = 0$ and y_1, y_2 and y_3 are not in the basis, we have found a feasible ordered basis for the original problem $B = (1, 3, 4)$.
- We replace the top row with $[-\pi^T \mathbf{b} | -\pi^T]$, where $\pi^T = \mathbf{c}_B^T \mathbf{A}_B^{-1}$ is computed using c^T from the original LP (1).
- We compute (note the order $\mathbf{c}_B^T = [c_1, c_3, c_4] = [-19, -12, -17]$ must match the order in the basis heading x_1, x_3, x_4)

$$\pi^T = \mathbf{c}_B^T \mathbf{A}_B^{-1} = [-19,\ -12,\ -17] \begin{pmatrix} 1 & 2 & -1 \\ 0 & 4 & -1 \\ -1 & -5 & 2 \end{pmatrix} = [-2,\ -1,\ -3].$$

- Then we compute $\pi^T \mathbf{b} = [-2, -1, -3][255, 117, 420]^T = -1827$.
- Hence, our CARRY-6 is

CARRY-6	$-z$	1827	2	1	3
	x_1	39	1	2	-1
	x_3	48	0	4	-1
	x_4	30	-1	-5	2

CARRY-6	$-z$	1827	2	1	3
	x_1	39	1	2	-1
	x_3	48	0	4	-1
	x_4	30	-1	-5	2

- The only variable not in the basis is x_2, so we compute $\bar{c}_2 = c_2 - \pi^T A_2 = -13 + [2, 1, 3][2, 1, 3]^T = 1 \geq 0$
- Since it is nonnegative, we conclude that the optimal value is -1827 attained at $x = [39,\ 0,\ 48,\ 30]^T$.