Primal-dual example

Suppose we are given the problem **P**:

$$\text{Minimize } z = x_1 + 3x_2 + 3x_3 + x_4$$

subject to

$$\begin{cases} 3x_1 & +4x_2 & -3x_3 & +x_4 & = & 2, \\ 3x_1 & -2x_2 & +6x_3 & -x_4 & = & 1, \\ 6x_1 & +4x_2 & & +x_4 & = & 4 \\ x_1, & x_2, & x_3, & x_4 & \geq & 0. \end{cases} \tag{1}$$

The dual **D** of **P** is the following:

$$\text{Maximize } w = 2\pi_1 + \pi_2 + 4\pi_3$$

subject to

$$\begin{cases} 3\pi_1 & +3\pi_2 & +6\pi_3 & \leq & 1, \\ 4\pi_1 & -2\pi_2 & +4\pi_3 & \leq & 3, \\ -3\pi_1 & +6\pi_2 & & \leq & 3, \\ \pi_1 & -\pi_2 & +\pi_3 & \leq & 1. \end{cases} \tag{2}$$

- Someone tells us that the vector $\pi = (\frac{1}{3},\, 0,\, 0)^T$ is an optimal vector for **D**.
- We compute that π is feasible and J, the indices of the *admissible* columns, is $\{1\}$.
- Complementary slackness implies that if π is optimal for **D**, then there exists a solution to **P** such that the only non-zero entry is x_1.
- We try to find it by solving the following *restricted primal problem* **RP1** using revised simplex.

$$\text{Minimize } \xi = x_1^r + x_2^r + x_3^r$$

subject to

$$\begin{cases} 3x_1 & +x_1^r & & & = & 2, \\ 3x_1 & & +x_2^r & & = & 1, \\ 6x_1 & & & +x_3^r & = & 4, \\ x_1, & x_1^r, & x_2^r, & x_3^r & \geq & 0. \end{cases}$$

- The cost vector we will use is for the restricted primal problem, i.e. x_1^r, x_2^r, and x_3^r each has cost 1 and x_1, x_2, x_3 and x_4 each has cost 0.
- We use the label π^r instead of π when solving **RP**.
- We start with x_1^r, x_2^r, and x_3^r in the basis, so $(\pi^r)^T = \mathbf{c}_B^T \mathbf{A}_B^{-1} = [1, 1, 1]$ and $(\pi^r)^T \mathbf{b} = 7$, and the initial CARRY matrix is:

$-\xi$	-7	-1	-1	-1
x_1^r	2	1	0	0
x_2^r	1	0	1	0
x_3^r	4	0	0	1

- There is only one column we can bring into the basis, the column associated with x_1. The relative cost of this column is $0 - (\pi^r)^T \mathbf{A}_1 = -12$.
- We compute that $\mathbf{A}_B^{-1}\mathbf{A}_1 = [3, 3, 6]^T$, and append $[-12, 3, 3, 6]^T$ to the tableau and pivot on the second row.

- We get the following CARRY, and we are done since x_2^r just left the basis, so it has non-negative relative cost, and we cannot pivot on the x_2, x_3 or x_4 columns since they are not admissible.

$-\xi$	-3	-1	3	-1
x_1^r	1	1	-1	0
x_1	$1/3$	0	$1/3$	0
x_3^r	2	0	-2	1

- Since the optimal value of **RP** is $\xi = 3$, we know that $\pi = (1/3,\ 0,\ 0)^T$ is NOT optimal for **D**.
- But we can use π^r to improve π. Our new π^* will have the form

$$\pi^* = \pi + \theta\pi^r. \tag{3}$$

- Here θ is a positive number that we will compute and π^r is an optimal vector in the dual **DRP1** to **RP1** which is:

$$\text{Maximize } w^r = 2\pi_1^r + \pi_2^r + 4\pi_3^r$$

$$\text{subject to} \quad \begin{cases} 3\pi_1^r & +3\pi_2^r & +6\pi_3^r & \leq & 0, \\ \pi_1^r & & & \leq & 1, \\ & \pi_2^r & & \leq & 1, \\ & & \pi_3^r & \leq & 1. \end{cases}$$

- We extract $(\pi^r)^T = [1, -3, 1]$ from the last CARRY matrix.
- Now we choose θ as large as possible so that the vector $(\pi^*)^T = (\frac{1}{3}, 0, 0) + \theta(1, -3, 1)$ is feasible for **D**,
- That is, we need $\pi^T\mathbf{A}_j + \theta(\pi^r)^T\mathbf{A}_j \leq c_j$ for every $j \in [n]$.
- Since π^r is feasible for **DRP**, $(\pi^r)^T\mathbf{A}_j \leq 0$, for every $j \in J$.
- Therefore, $\theta = \min\limits_{\substack{j \notin J, \\ (\pi^r)^T\mathbf{A}_j > 0}} \left\{ \dfrac{c_j - \pi^T\mathbf{A}_j}{(\pi^r)^T\mathbf{A}_j} \right\}$
- We compute $(\pi^r)^T\mathbf{A}_2 = 14$, $(\pi^r)^T\mathbf{A}_3 = -21$, and $(\pi^r)^T\mathbf{A}_4 = 5$, so
- $\theta = \min\{\frac{c_2 - \pi^T\mathbf{A}_2}{14}, \frac{c_4 - \pi^T\mathbf{A}_4}{5}\} = \min\{\frac{5/3}{14}, \frac{2/3}{5}\} = \frac{5}{42}$, and
- $(\pi^*)^T = \pi^T + \theta(\pi^\mathbf{r})^T = (\frac{1}{3}, 0, 0)^T + \frac{5}{42}(1, -3, 1)^T = (\frac{19}{42}, \frac{-5}{14}, \frac{5}{42})^T$
- Note that $(\pi^*)^T\mathbf{b} > \pi^T\mathbf{b}$ and set $\pi = \pi^*$.

- We compute that $J = \{1, 2\}$
- Note that, by 5.3, 1 must be in J, because it was in the basis at the end of the last iteration
- Our last carry matrix was

$-\xi$	-3	-1	3	-1
x_1^r	1	1	-1	0
x_1	$1/3$	0	$1/3$	0
x_3^r	2	0	-2	1

- The relative cost of x_2 is $0 - (\pi^r)^T\mathbf{A}_2 = -14$, so we pivot on the x_2 column.
- We compute that $\mathbf{A}_B^{-1}\mathbf{A}_2 = [6, -2/3, 8]^T$, append $[-14, 6, -2/3, 8]^T$ to the CARRY matrix and pivot on the first row to obtain.

$-\xi$	$-2/3$	$4/3$	$2/3$	-1
x_2	$1/6$	$1/6$	$-1/6$	0
x_1	$4/9$	$1/9$	$2/9$	0
x_3^r	$2/3$	$-4/3$	$-2/3$	1

- The relative cost of x_2^r is $1 - (\pi^r)^T\mathbf{e_2} = 1 + \frac{2}{3} = \frac{5}{3} \geq 0$, so we are done, because x_1^r just left the basis and the x_3 and x_4 columns are not admissible.

- We extract $(\pi^r)^T = [-\frac{4}{3}, \frac{-2}{3}, 1]$ from the CARRY matrix
- For $j \notin J$, we compute $(\pi^r)^T \mathbf{A}_j$, and $(\pi^r)^T \mathbf{A}_3 = 0$, and $(\pi^r)^T \mathbf{A}_4 = 1/3$.
- Hence $\theta = \min\{\frac{c_4 - \pi^T \mathbf{A}_4}{1/3}\} = \min\{\frac{1-13/14}{1/3}\} = \frac{3}{14}$
- So $(\pi^*)^T = \pi^T + \frac{3}{14}(\pi^r)^T = [\frac{1}{6}, \frac{-1}{2}, \frac{1}{3}]$.
- We set $\pi^T = (\pi^*)^T$ and compute $J = \{1, 2, 4\}$. 5.3 implies that $\{1, 2\} \subseteq J$, because x_1 and x_2 were in the basis at the end of the last iteration
- Recall that our last CARRY matrix was

$$\begin{array}{r|c|c|c|c|}
\hline
-\xi & -2/3 & 4/3 & 2/3 & -1 \\
\hline
x_2 & 1/6 & 1/6 & -1/6 & 0 \\
\hline
x_1 & 4/9 & 1/9 & 2/9 & 0 \\
\hline
x_3^r & 2/3 & -4/3 & -2/3 & 1 \\
\hline
\end{array}$$

- The relative cost of x_4 is $0 - (\pi^r)^T = \frac{-1}{3} < 0$, so we pivot on column 4
- $\mathbf{A}_B^{-1}\mathbf{A}_4 = [\frac{1}{3}, \frac{-1}{9}, \frac{1}{3}]^T$, so we append $[\frac{-1}{3}, \frac{1}{3}, \frac{-1}{9}, \frac{1}{3}]^T$ to the CARRY matrix and pivot on row 1, to obtain

$$\begin{array}{r|c|c|c|c|}
\hline
-\xi & -1/2 & 3/2 & 1/2 & -1 \\
\hline
x_4 & 1/2 & 1/2 & -1/2 & 0 \\
\hline
x_1 & 1/2 & 1/6 & 1/6 & 0 \\
\hline
x_3^r & 1/2 & -3/2 & -1/2 & 1 \\
\hline
\end{array}$$

- x_2 just left the basis and the relative cost of x_1^r is $1 + 3/2 = 5/2$ and the relative cost of x_2^r is $1 + 1/2 = 3/2$, so we are done.
- π^T is not optimal because $\xi_{opt} = 1/2 > 0$.

- We extract $(\pi^r)^T = [-3/2, -1/2, 1]$ from the CARRY matrix.
- For $j \notin J$, we compute $(\pi^r)^T \mathbf{A}_j$, and we have that $(\pi^r)^T \mathbf{A}_3 = 3/2$.
- Hence $\theta = \min\{\frac{c_3 - \pi^T \mathbf{A}_3}{3/2}\} = \min\{\frac{3-(-7/2)}{3/2}\} = \frac{13}{3}$
- We compute $(\pi^*)^T = \pi^T + \frac{13}{3}(\pi^r)^T = [-\frac{19}{3}, \frac{-8}{3}, \frac{14}{3}]$ and set $\pi^T = (\pi^*)^T$, and $J = \{1, 3, 4\}$
- Recall that our last CARRY matrix was

$$\begin{array}{r|c|c|c|c|}
\hline
-\xi & -1/2 & 3/2 & 1/2 & -1 \\
\hline
x_4 & 1/2 & 1/2 & -1/2 & 0 \\
\hline
x_1 & 1/2 & 1/6 & 1/6 & 0 \\
\hline
x_3^r & 1/2 & -3/2 & -1/2 & 1 \\
\hline
\end{array}$$

- The relative cost x_3 is $0 - (\pi^r)^T \mathbf{A}_3 = -3/2$, so we pivot on column 3
- We compute $\mathbf{A}_B^{-1}\mathbf{A}_3 = [-9/2, 1/2, 3/2]^T$, so we append $[-3/2, -9/2, 1/2, 3/2]^T$ to the CARRY matrix and pivot on row 3, to obtain

$$\begin{array}{r|c|c|c|c|}
\hline
-\xi & 0 & 0 & 0 & 0 \\
\hline
x_4 & 2 & -4 & -2 & 3 \\
\hline
x_1 & 1/3 & 2/3 & 1/3 & -1/3 \\
\hline
x_3 & 1/3 & -1 & -1/3 & 2/3 \\
\hline
\end{array}$$

- Since $\xi_{opt} = 0$, $(-\frac{19}{3}, \frac{-8}{3}, \frac{14}{3})^T$ is an optimal solution for $\mathbf{D}$ and the optimal solution $\mathbf{P}$ is $(\frac{1}{3}, 0, \frac{1}{3}, 2)^T$. The cost is $\frac{10}{3}$.

The following table summaries the entire process.

Iteration	π^T	$\pi^T \mathbf{b}$	J	$(\pi^r)^T$	$(\pi^r)^T \mathbf{b}$	θ
1	$[\frac{1}{3}, 0, 0]$	$\frac{2}{3}$	$\{1\}$	$[1, -3, 1]$	3	$\frac{5}{42}$
2	$[\frac{19}{42}, \frac{-5}{14}, \frac{5}{42}]$	$\frac{43}{42}$	$\{1, 2\}$	$[-\frac{4}{3}, -\frac{2}{3}, 1]$	$\frac{2}{3}$	$\frac{3}{14}$
3	$[\frac{1}{6}, -\frac{1}{2}, \frac{1}{3}]$	$\frac{7}{6}$	$\{1, 2, 4\}$	$[-\frac{3}{2}, -\frac{1}{2}, 1]$	$\frac{1}{2}$	$\frac{13}{3}$
4	$[-\frac{19}{3}, -\frac{8}{3}, \frac{14}{3}]$	$\frac{10}{3}$	$\{1, 3, 4\}$	$[0, 0, 0]$	0	N/A