Suppose we are given the problem

$$\text{Minimize } z = 2x_1 + 3x_2 + 4x_3 + 5x_4$$

subject to

$$\begin{cases} x_1 & -x_2 & +x_3 & -x_4 & \geq & 10, \\ x_1 & -2x_2 & +3x_3 & -4x_4 & \geq & 6, \\ 3x_1 & -4x_2 & +5x_3 & -6x_4 & \geq & 15 \\ x_1, & x_2, & x_3, & x_4 & \geq & 0. \end{cases} \tag{1}$$

- Do we add slack variables or surplus variables?
- Do we automatically have a basic feasible solution after adding surplus variables?
- We could do two phase simplex,
- but since the coefficients in the objective function are positive, we can use dual simplex.

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-z$	0	2	3	4	5	0	0	0
x_5	-10	-1	1	-1	1	1	0	0
x_6	-6	-1	2	-3	4	0	1	0
x_7	-15	-3	4	-5	6	0	0	1

- After adding surplus variables and solving for the basis $B = (5, 6, 7)$ we have the tableau above.
- B is **not** a (primal) feasible basis, because $\mathbf{A}_B^{-1}\mathbf{b} = [-10, -6, -15]^T$ has negative entries. It is a dual feasible basis, because $\bar{\mathbf{c}}^T \geq \mathbf{0}^T$, so we can use the dual simplex algorithm.
- In dual simplex, we **first** pick the pivot **row** by selecting a row with a negative entry in column 0. We arbitrarily select row $r = 1$.
- We now pick the pivot column so that $z = -a_{0,0}$ does not decrease and the basis remains dual feasible.
- Since we do **not** want $z = -a_{0,0}$ to decrease, we pick the pivot column s so that $a_{r,s} < 0$.
- Since we want $\bar{\mathbf{c}}^T$ to remain non-negative we pick the x_1 column, because $\frac{a_{0,1}}{a_{r,1}} = \frac{2}{-1} > \frac{4}{-1} = \frac{a_{0,3}}{a_{r,3}}$.
- **Rule**: Pick column s so that $\frac{a_{0,s}}{a_{r,s}}$ is maximized subject to $a_{r,s} < 0$.

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-z$	-20	0	5	2	7	2	0	0
x_1	10	1	-1	1	-1	-1	0	0
x_6	4	0	1	-2	3	-1	1	0
x_7	15	0	1	-2	3	-3	0	1

- Now every $a_{i,0}$ for $i \in [m]$ is nonnegative. So, the tableau is optimal.
- But suppose that the boss adds a new restriction:

$$x_1 + 2x_2 + 3x_3 - 4x_4 \leq 8.$$

- With the dual simplex, we do not need to start from scratch. We simply add the new row and one more column to our final tableau.

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
$-z$	-20	0	5	2	7	2	0	0	0
x_1	10	1	-1	1	-1	-1	0	0	0
x_6	4	0	1	-2	3	-1	1	0	0
x_7	15	0	1	-2	3	-3	0	1	0
x_8	8	1	2	3	-4	0	0	0	1

- This tableau has a new row for the new equation and also a new slack variable
- We have to make sure that this tableau is solved for the basis, so we must exclude the basic variables from the new row.

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
$-z$	-20	0	5	2	7	2	0	0	0
x_1	10	1	-1	1	-1	-1	0	0	0
x_6	4	0	1	-2	3	-1	1	0	0
x_7	15	0	1	-2	3	-3	0	1	0
x_8	-2	0	3	2	-3	1	0	0	1

- We select row 4 as the pivot row because $a_{4,0} = -2 < 0$. We then pivot on column x_4 because $a_{4,4} = -3$ is the only negative entry in row 4.

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
$-z$	$-74/3$	0	12	$20/3$	0	$13/3$	0	0	$7/3$
x_1	$32/3$	1	-2	$1/3$	0	$-4/3$	0	0	$-1/3$
x_6	2	0	4	0	0	0	1	0	1
x_7	13	0	4	0	0	-2	0	1	1
x_4	$2/3$	0	-1	$-2/3$	1	$-1/3$	0	0	$-1/3$

- This is the final tableau because the basis is now both primal and dual feasible.

- Here are the constraints of the current problem (without the non-negativity constraints).

$$\begin{cases} x_1 - x_2 + x_3 - x_4 \geq 10, \\ x_1 - 2x_2 + 3x_3 - 4x_4 \geq 6, \\ 3x_1 - 4x_2 + 5x_3 - 6x_4 \geq 15 \\ -x_1 - 2x_2 - 3x_3 + 4x_4 \geq -8 \end{cases}$$

- Using the final basis $B = (1, 6, 7, 4)$ we can compute an optimum for the dual $(\pi^*)^T = \mathbf{c}_B^T \mathbf{A}_B^{-1} = \begin{pmatrix} \frac{13}{3} & 0 & 0 & \frac{7}{3} \end{pmatrix}$
- If we change $\mathbf{b}$ to $\mathbf{b}'$, then B remains a dual feasible basis. Why? (The constraints of the dual are not changed)
- Assume that changing b_1 from 10 to 12 or changing b_4 from -8 to -6 will not affect primal feasiblilty, i.e. $\mathbf{A}_B^{-1}\mathbf{b}' \geq \mathbf{0}$ for either change.
- Which change should we make?
- If $\mathbf{x}^{**}$ is the basic solution corresponding to B for the modified LP, then $\mathbf{x}^{**}$ and π^* are feasible for the modified LP and its dual, resp., and $\mathbf{c}^T\mathbf{x}^{**} = \mathbf{c}_B^T\mathbf{x}_B^{**} = \mathbf{c}_B^T\left(\mathbf{A}_B^{-1}\mathbf{b}'\right) = \left(\mathbf{c}_B^T\mathbf{A}_B^{-1}\right)\mathbf{b}' = (\pi^*)^T\mathbf{b}' = \pi_1^* b_1' + \pi_2^* b_2' + \pi_3^* b_3' + \pi_4^* b_4'$
- b_1 from 10 to 12 $\implies$ cost increases by $2\pi_1^* = \frac{26}{3}$
- b_4 from -8 to -6 $\implies$ cost increase by $2\pi_4^* = \frac{14}{3}$. ✓
- We can think of π_i^* as the *shadow price* or *marginal price* of the resource associated with constraint i.

- What if we change $\mathbf{b}$ to $\mathbf{b}' = \begin{pmatrix} 10 & 6 & 15 & -5 \end{pmatrix}^T$?
- We have that $\mathbf{A}_B^{-1}\mathbf{b} = \begin{pmatrix} \frac{35}{3} & -1 & 10 & \frac{5}{3} \end{pmatrix}^T$, so $B = (1, 6, 7, 4)$ is no longer primal feasible. It is still dual feasible.
- We can use dual simplex by replacing entries $a_{0,0}$ to $a_{m,0}$ in the final tableau. Replace $a_{0,0}$ with $-\mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{b}' = -(\pi^*)^T\mathbf{b}' = -\frac{74}{3} - 3\pi_4^* = -\frac{95}{3}$ and replace $\begin{pmatrix} a_{0,1} & \dots & a_{0,m} \end{pmatrix}^T$ with $\mathbf{A}_B^{-1}\mathbf{b}'$.
- So we start dual simplex with the following tableau

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
$-z$	$-95/3$	0	12	20/3	0	13/3	0	0	7/3
x_1	35/3	1	-2	1/3	0	$-4/3$	0	0	$-1/3$
x_6	-1	0	4	0	0	0	1	0	1
x_7	10	0	4	0	0	-2	0	1	1
x_4	5/3	0	-1	$-2/3$	1	$-1/3$	0	0	$-1/3$

- This LP is infeasible, because row 2 corresponds to the equation $-1 = 4x_2 + x_6 + x_8$ which has not solution with $x_2, x_6, x_8 \geq 0$.
- What does this imply about the dual? It is unbounded because π^* is a feasible solution for the dual and the primal is infeasible.