Example (2.7 from book)

- We will use the following *pivot rules*. *Pivot rules* are additional rules for selecting the pivot row and the pivot column.
- Select the pivot column s so that $a_{0,s} = \overline{c}_s \leq \overline{c}_j = a_{0,j}$ for all $j \in [n]$. (In this example, this always gives a unique choice.) This was Dantzig's original rule for selecting a pivot column and is sometime called *Dantzig's rule* or the *largest coefficient rule*.
- In the case of ties when selecting the pivot row, select the row so that the smallest index leaves the basis. In other words, assuming the tableau is solved for the basis $B = (j_1, \ldots, j_m)$, select the pivot row r, from all valid choices, so that j_r is as small as possible. This always gives a unique choice.

Tableau 1

$$T_1 = \begin{array}{c|c|ccccccc} & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ \hline -z & 3 & -\frac{3}{4} & 20 & -\frac{1}{2} & 6 & 0 & 0 & 0 \\ \hline x_5 & 0 & 1/4 & -8 & -1 & 9 & 1 & 0 & 0 \\ x_6 & 0 & \frac{1}{2} & -12 & -\frac{1}{2} & 3 & 0 & 1 & 0 \\ x_7 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}$$

bfs $(0,0,0,0,0,0,1)^T$, basis $(5,6,7)$.

1. The bfs is degenerate.
2. x_5 and x_6 are basic variables at zero-level
3. We will bring x_1 into the basis (replacing x_5) at zero-level.

Tableau 2

$$T_1 = \begin{array}{c|c|ccccccc} & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ \hline -z & 3 & -\frac{3}{4} & 20 & -\frac{1}{2} & 6 & 0 & 0 & 0 \\ \hline x_5 & 0 & \frac{1}{4} & -8 & -1 & 9 & 1 & 0 & 0 \\ x_6 & 0 & \frac{1}{2} & -12 & -\frac{1}{2} & 3 & 0 & 1 & 0 \\ x_7 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}$$

bfs $(0,0,0,0,0,0,1)^T$, basis $(5,6,7)$.

$$T_2 = \begin{array}{c|c|ccccccc} & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ \hline -z & 3 & 0 & -4 & -\frac{7}{2} & 33 & 3 & 0 & 0 \\ \hline x_1 & 0 & 1 & -32 & -4 & 36 & 4 & 0 & 0 \\ x_6 & 0 & 0 & 4 & \frac{3}{2} & -15 & -2 & 1 & 0 \\ x_7 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}$$

bfs $(0,0,0,0,0,0,1)^T$, basis $(1,6,7)$.

Tableau 3

$$T_1 = \begin{array}{c|c|ccccccc} & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ \hline -z & 3 & -\frac{3}{4} & 20 & -\frac{1}{2} & 6 & 0 & 0 & 0 \\ \hline x_5 & 0 & \frac{1}{4} & -8 & -1 & 9 & 1 & 0 & 0 \\ x_6 & 0 & \frac{1}{2} & -12 & -\frac{1}{2} & 3 & 0 & 1 & 0 \\ x_7 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}$$

bfs $(0,0,0,0,0,0,1)^T$, basis $(5,6,7)$.

$$T_3 = \begin{array}{c|c|ccccccc} & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ \hline -z & 3 & 0 & 0 & -2 & 18 & 1 & 1 & 0 \\ \hline x_1 & 0 & 1 & 0 & 8 & -84 & -12 & 8 & 0 \\ x_2 & 0 & 0 & 1 & \frac{3}{8} & -\frac{15}{4} & -\frac{1}{2} & \frac{1}{4} & 0 \\ x_7 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}$$

bfs $(0,0,0,0,0,0,1)^T$, basis $(1,2,7)$.

Tableau 4

$$T_1 = \begin{array}{c|c|ccccccc} & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ \hline -z & 3 & -\frac{3}{4} & 20 & -\frac{1}{2} & 6 & 0 & 0 & 0 \\ \hline x_5 & 0 & \frac{1}{4} & -8 & -1 & 9 & 1 & 0 & 0 \\ x_6 & 0 & \frac{1}{2} & -12 & -\frac{1}{2} & 3 & 0 & 1 & 0 \\ x_7 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}$$

bfs $(0,0,0,0,0,0,1)^T$, basis $(5,6,7)$.

$$T_4 = \begin{array}{c|c|ccccccc} & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ \hline -z & 3 & \frac{1}{4} & 0 & 0 & -3 & -2 & 3 & 0 \\ \hline x_3 & 0 & \frac{1}{8} & 0 & 1 & -\frac{21}{2} & -\frac{3}{2} & 1 & 0 \\ x_2 & 0 & -\frac{3}{64} & 1 & 0 & \frac{3}{16} & \frac{1}{16} & -\frac{1}{8} & 0 \\ x_7 & 1 & -\frac{1}{8} & 0 & 0 & \frac{21}{2} & \frac{3}{2} & -1 & 1 \end{array}$$

bfs $(0,0,0,0,0,0,1)^T$, basis $(3,2,7)$.

Tableau 5

$$T_1 = \begin{array}{c|c|ccccccc} & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ \hline -z & 3 & -\frac{3}{4} & 20 & -\frac{1}{2} & 6 & 0 & 0 & 0 \\ \hline x_5 & 0 & \frac{1}{4} & -8 & -1 & 9 & 1 & 0 & 0 \\ x_6 & 0 & \frac{1}{2} & -12 & -\frac{1}{2} & 3 & 0 & 1 & 0 \\ x_7 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}$$

bfs $(0,0,0,0,0,0,1)^T$, basis $(5,6,7)$.

$$T_5 = \begin{array}{c|c|ccccccc} & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ \hline -z & 3 & -\frac{1}{2} & 16 & 0 & 0 & -1 & 1 & 0 \\ \hline x_3 & 0 & -\frac{5}{2} & 56 & 1 & 0 & 2 & -6 & 0 \\ x_4 & 0 & -\frac{1}{4} & \frac{16}{3} & 0 & 1 & \frac{1}{3} & -\frac{2}{3} & 0 \\ x_7 & 1 & \frac{5}{2} & -56 & 0 & 0 & -2 & 6 & 1 \end{array}$$

bfs $(0,0,0,0,0,0,1)^T$, basis $(3,4,7)$.

Tableau 6

$$T_1 = \begin{array}{c|c|ccccccc}
 & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\
\hline
-z & 3 & -\frac{3}{4} & 20 & -\frac{1}{2} & 6 & 0 & 0 & 0 \\
\hline
x_5 & 0 & \frac{1}{4} & -8 & -1 & 9 & 1 & 0 & 0 \\
x_6 & 0 & \frac{1}{2} & -12 & -\frac{1}{2} & 3 & 0 & 1 & 0 \\
x_7 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}$$

bfs $(0,0,0,0,0,0,1)^T$, basis $(5,6,7)$.

$$T_6 = \begin{array}{c|c|ccccccc}
 & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\
\hline
-z & 3 & -\frac{7}{4} & 44 & \frac{1}{2} & 0 & 0 & -2 & 0 \\
\hline
x_5 & 0 & -\frac{5}{4} & 28 & \frac{1}{2} & 0 & 1 & -3 & 0 \\
x_4 & 0 & \frac{1}{6} & -4 & -\frac{1}{6} & 1 & 0 & \frac{1}{3} & 0 \\
x_7 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}$$

bfs $(0,0,0,0,0,0,1)^T$, basis $(5,4,7)$.

Tableau 7 same as Tableau 1!

$$T_1 = \begin{array}{c|c|ccccccc}
 & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\
\hline
-z & 3 & -\frac{3}{4} & 20 & -\frac{1}{2} & 6 & 0 & 0 & 0 \\
\hline
x_5 & 0 & \frac{1}{4} & -8 & -1 & 9 & 1 & 0 & 0 \\
x_6 & 0 & \frac{1}{2} & -12 & -\frac{1}{2} & 3 & 0 & 1 & 0 \\
x_7 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}$$

bfs $(0,0,0,0,0,0,1)^T$, basis $(5,6,7)$.

$$T_7 = \begin{array}{c|c|ccccccc}
 & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\
\hline
-z & 3 & -\frac{3}{4} & 20 & -\frac{1}{2} & 6 & 0 & 0 & 0 \\
\hline
x_5 & 0 & \frac{1}{4} & -8 & -1 & 9 & 1 & 0 & 0 \\
x_6 & 0 & \frac{1}{2} & -12 & -\frac{1}{2} & 3 & 0 & 1 & 0 \\
x_7 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}$$

bfs $(0,0,0,0,0,0,1)^T$, basis $(5,6,7)$.

Example with Lexicographic simplex

- We start with the same tableau as example 2.7 from the book, but this time we following the lexicographic simplex method.

Tableau 1

$T_1 =$

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-z$	3	$-\frac{3}{4}$	20	$-\frac{1}{2}$	6	0	0	0
x_5	0	$\frac{1}{4}$	-8	-1	9	1	0	0
x_6	0	$\frac{1}{2}$	-12	$-\frac{1}{2}$	3	0	1	0
x_7	1	0	0	1	0	0	0	1

- Note that the rows 1 through 3 are lexicographically positive.
- We pivot on column 1 (we could have also picked column 3).
- Rows 1 and 2 are tied by the standard simplex pivot row selection rule.
- But row 1 divided by $a_{1,1}$ is lexicographically less that row 2 divided by $a_{2,1}$, so we pick row 1.

Tableau 2

$T_2 =$

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-z$	3	0	-4	$-\frac{7}{2}$	33	3	0	0
x_1	0	1	-32	-4	36	4	0	0
x_6	0	0	4	$\frac{3}{2}$	-15	-2	1	0
x_7	1	0	0	1	0	0	0	1

- Note that the top row has increased lexicographically
- We pivot on column 2.
- There is no choice for the pivot row, we must pick row 2.

Tableau 3

$T_3 =$

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-z$	3	0	0	-2	18	1	1	0
x_1	0	1	0	8	-84	-12	8	0
x_2	0	0	1	$\frac{3}{8}$	$-\frac{15}{4}$	$-\frac{1}{2}$	$\frac{1}{4}$	0
x_7	1	0	0	1	0	0	0	1

- We select 3 as the pivot column, it is the only possible pivot column
- In this step, rows 1 and 2 are tied, and we select row 2 because row 2 divided by $a_{2,3}$ is lexicographically less than row 1 divided by $a_{1,3}$.

Tableau 4

$$T_4 = \begin{array}{c|c|ccccccc}
 & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\
\hline
-z & 3 & 0 & 16/3 & 0 & -2 & -5/3 & 7/3 & 0 \\
\hline
x_1 & 0 & 1 & -64/3 & 0 & -4 & -4/3 & 8/3 & 0 \\
x_3 & 0 & 0 & 8/3 & 1 & -10 & -4/3 & 2/3 & 0 \\
x_7 & 1 & 0 & -8/3 & 0 & 10 & 4/3 & -2/3 & 1
\end{array}$$

- We pick column 5 as our pivot column.
- Once we select column 5 as our pivot column, we must pivot on row 3

Tableau 5

$$T_5 = \begin{array}{c|c|ccccccc}
 & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\
\hline
-z & 17/4 & 0 & 2 & 0 & 21/2 & 0 & 3/2 & 5/4 \\
\hline
x_1 & 1 & 1 & -24 & 0 & 6 & 0 & 2 & 1 \\
x_3 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
x_5 & 3/4 & 0 & -2 & 0 & 15/2 & 1 & -1/2 & 3/4
\end{array}$$

- Now we cannot select a pivot column.
- The bfs $(1, 0, 1, 0, 3/4, 0)^T$ with associated basis $(1, 3, 5)$ is the optimal solution, because the entries in the top row (columns 1 through 7) are non-negative. In other words, the relative cost vector $\bar{\mathbf{c}}^T$ is non-negative.

Quiz

- Find the pivot entry using Bland's rule and lexicographic simplex

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-z$	8	2	0	0	2	-3	0	0
x_7	2	2	0	0	3	4	0	1
x_6	6	10	0	0	4	12	1	0
x_3	4	5	0	1	2	8	0	0
x_2	2	0	1	0	4	3	0	0

- Using Bland's rule we select column x_5 and row 3
- Using lexicographic simplex we select column x_5 and row 1