Homework 2
M588, 2015 SPRING
DUE: March 4 (W)

NAME:..
SCORE:..

1. We say that a step in the simplex method is *degenerate* if the cost function stays fixed. In this question you show that it is sometimes necessary to take degenerate steps. That is, show that there is a simplex tableau whose corresponding basic feasible solution is not optimal such that no matter which possible pivot column is selected the step is degenerate.

2. Show that an LP cannot cycle unless there exists a basic feasible solution $\boldsymbol{x}$ such that $x_j = 0$ for at least $n - m + 2$ indices $j \in \{1, \ldots, n\}$.

3. Consider the following LP, which we call Problem P:

$$\begin{array}{lllll} \min & x_1 & & +x_3 & \\ & x_1 & +2x_2 & & \leq 5 \\ & & x_2 & +2x_3 & = 6 \\ & x_1 & , 2x_3 & , x_3 & \geq 0 \end{array}$$

   (a) Solve P by the simplex algorithm.
   (b) Write the dual of P.
   (c) Write the complementary slackness conditions for this problem and use them to solve the dual D. Check your answer by evaluating the optimal cost of P and D.

4. Prove Lemma 8.10 from the book, i.e., prove that for any $\boldsymbol{a} \in \mathbb{R}^n$, there exists a matrix R such that $RR^T = I$ and $R\boldsymbol{a} = (\|\boldsymbol{a}\|, 0, \ldots, 0)^T$.

5. Prove the following variant of Farkas lemma. You can use either the strong duality theorem or the variant of Farkas lemma that is the book (Theorem 3.5). There exists $\boldsymbol{x} \in \mathbb{R}^n$ such that $A\boldsymbol{x} \leq \boldsymbol{b}$ if and only if for all $\boldsymbol{y} \in \mathbb{R}^m$ if $\boldsymbol{y} \geq \boldsymbol{0}$ and $\boldsymbol{y}^T A = \boldsymbol{0}^T$ then $\boldsymbol{y}^T \boldsymbol{b} \geq 0$.

6. **I don't care about the $-8L$ and $-9L$ here. If you can prove it for some $-c_1L$ and $-c_2L$, that is OK.**

Use the variant of Farkas Lemma proved in the previous problem to prove the following. For any $A \in \mathbb{Q}^{m\times n}$ and $\boldsymbol{b} \in \mathbb{Q}^m$, let $L = \langle A\rangle + \langle b\rangle$, $\eta = 2^{-8L}$, $\varepsilon = 2^{-9L}$ and $P := \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} \le \boldsymbol{b} + \eta\mathbf{1}\}$. For any $\boldsymbol{x}$ such that $A\boldsymbol{x} \le \boldsymbol{b}$, we have that $P \supseteq B(\boldsymbol{x}, \varepsilon)$ and if $A\boldsymbol{x} \le \boldsymbol{b}$ has no solutions then $P = \emptyset$.

Hint: You may use the fact that for any matrix rational matrix B and any vector $\mathbf{c}$, if the system $B\boldsymbol{x} = \mathbf{c}$ has solution, then the system has a solution $\boldsymbol{x}$ whose entries have absolute value less than $2^{4(<B>+<\mathbf{c}>)}$, i.e. $\|\boldsymbol{x}\|_\infty \le 2^{4(<B>+<\mathbf{c}>)}$ and that the statement remain true if we require that a solution $\boldsymbol{x}$ such that $\boldsymbol{x} \ge \mathbf{0}$.

Notation: If $x \in \mathbb{Q}$ and $x = p/q$ where p and q are integers and p and q are relatively prime and $q > 0$, then the size of x is

$$\langle x\rangle := 1 + \lceil \log_2 |p| + 1\rceil + \lceil \log_2 q + 1\rceil\,.$$

If $b \in \mathbb{Q}^n$ and $A \in Q^{m\times n}$, then $\langle b\rangle = m + \sum_{i=1}^n \langle b_i\rangle$, $\langle A\rangle = nm + \sum_{i=1}^m \sum_{j=1}^n \langle a_{i,j}\rangle$.