Two-phase simplex

Suppose we are given the linear program

$$\text{Maximize } z = x_1 - x_2 + x_3$$

$$\begin{cases} 2x_1 & -x_2 & +2x_3 & +x_4 & & & = & 4 \\ 2x_1 & -3x_2 & +x_3 & & +x_5 & & = & -5 \\ -x_1 & +x_2 & -2x_3 & & & +x_6 & = & -1 \\ x_1, & x_2, & x_3, & x_4, & x_5 & x_6 & \geq & 0. \end{cases}$$

- Find a basic feasible solution?

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- $(0, 0, 0, 4, -5, -1)^T$ is not a bfs!!!!! It has negative entries.

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- $(0, 0, 0, 4, -5, -1)^T$ is not a bfs!!!!! It has negative entries.
- It is not obvious how to find one or that a feasible solution even exists.

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- Find a basic feasible solution?
- $(0, 0, 0, 4, -5, -1)^T$ is not a bfs!!!!! It has negative entries.
- It is not obvious how to find one or that a feasible solution even exists.
- We need the two-phase simplex method!

First step in the two phase simplex method:
Make sure $b \geq 0$. How can we do this?

$$\text{Maximize } z = x_1 - x_2 + x_3$$

$$\begin{cases} 2x_1 & -x_2 & +2x_3 & +x_4 & & & = & 4 \\ 2x_1 & -3x_2 & +x_3 & & +x_5 & & = & -5 \\ -x_1 & +x_2 & -2x_3 & & & +x_6 & = & -1 \\ x_1, & x_2, & x_3, & x_4, & x_5 & x_6 & \geq & 0. \end{cases}$$

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Make sure $b \geq 0$. How can we do this?
Multiply row i by -1 if $b_i < 0$.
Note: We do not change the constraints by doing this.

$$\text{Maximize } z = x_1 - x_2 + x_3$$

$$\left\{\begin{array}{rrrrrrcr} 2x_1 & -x_2 & +2x_3 & +x_4 & & & = & 4 \\ 2x_1 & -3x_2 & +x_3 & & +x_5 & & = & -5 \\ -x_1 & +x_2 & -2x_3 & & & +x_6 & = & -1 \\ x_1, & x_2, & x_3, & x_4, & x_5 & x_6 & \geq & 0. \end{array}\right.$$

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$$\text{Maximize } z = x_1 - x_2 + x_3$$

$$\begin{cases} 2x_1 & -x_2 & +2x_3 & +x_4 & & & = & 4 \\ -2x_1 & +3x_2 & -x_3 & & -x_5 & & = & 5 \\ x_1 & -x_2 & +2x_3 & & & -x_6 & = & 1 \\ x_1, & x_2, & x_3, & x_4, & x_5 & x_6 & \geq & 0. \end{cases}$$

We now want a linear program that

Original LP

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We now want a linear program that

1. is already solved for a basis, and

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1. is already solved for a basis, and
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We now want a linear program that

1. is already solved for a basis, and
2. whose optimal solution will either
3. give us a basic feasible solution for the original LP, or
4. prove that the original LP was infeasible.

Original LP

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$$\begin{cases} 2x_1 & -x_2 & +2x_3 & +x_4 & & & = & 4 \\ -2x_1 & +3x_2 & -x_3 & & -x_5 & & = & 5 \\ x_1 & -x_2 & +2x_3 & & & -x_6 & = & 1 \\ x_1, & x_2, & x_3, & x_4, & x_5 & x_6 & \geq & 0. \end{cases}$$

We now want a linear program that

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Add y_1 to equation 2 and y_2 to equation 3.

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$$\left\{ \begin{array}{rrrrrrcr} 2x_1 & -x_2 & +2x_3 & +x_4 & & & = & 4 \\ -2x_1 & +3x_2 & -x_3 & & -x_5 & & = & 5 \\ x_1 & -x_2 & +2x_3 & & & -x_6 & = & 1 \\ x_1, & x_2, & x_3, & x_4, & x_5 & x_6 & \geq & 0. \end{array} \right.$$

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Add y_1 to equation 2 and y_2 to equation 3.

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$$\text{Maximize } \xi = -y_1 - y_2$$

$$\begin{cases} 2x_1 & -x_2 & +2x_3 & +x_4 & & & & & = & 4 \\ -2x_1 & +3x_2 & -x_3 & & -x_5 & & +y_1 & & = & 5 \\ x_1 & -x_2 & +2x_3 & & & -x_6 & & +y_2 & = & 1 \\ x_1, & x_2, & x_3, & x_4, & x_5, & x_6, & y_1, & y_2 & \geq & 0. \end{cases}$$

We now want a linear program that

1. is already solved for a basis, and
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$x_7 = y_1$ and $x_8 = y_2$, bfs $(0,0,0,4,0,0,5,1)^T$

		x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2
$-\xi$	0	0	0	0	0	0	0	-1	-1
x_4	4	2	-1	2	1	0	0	0	0
y_1	5	-2	3	-1	0	-1	0	1	0
y_2	1	1	-1	2	0	0	-1	0	1

- Can we start pivoting?

		x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2
$-\xi$	0	0	0	0	0	0	0	-1	-1
x_4	4	2	-1	2	1	0	0	0	0
y_1	5	-2	3	-1	0	-1	0	1	0
y_2	1	1	-1	2	0	0	-1	0	1

- Can we start pivoting?
- NO!, the coefficients of the basic variables y_1 and y_2 in row 0 are not zero.

		x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2
$-\xi$	0	0	0	0	0	0	0	-1	-1
x_4	4	2	-1	2	1	0	0	0	0
y_1	5	-2	3	-1	0	-1	0	1	0
y_2	1	1	-1	2	0	0	-1	0	1

- Can we start pivoting?
- NO!, the coefficients of the basic variables y_1 and y_2 in row 0 are not zero.
- Remove y_1 and y_2 from the top equation.

		x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2
$-\xi$	6	-1	2	1	0	-1	-1	0	0
x_4	4	2	-1	2	1	0	0	0	0
y_1	5	-2	3	-1	0	-1	0	1	0
y_2	1	1	-1	2	0	0	-1	0	1

bfs $(0, 0, 0, 4, 0, 0, 5, 1)$, basis $\{4, 7, 8\}$.

- Pick a pivot column.

		x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2
$-\xi$	6	-1	2	1	0	-1	-1	0	0
x_4	4	2	-1	2	1	0	0	0	0
y_1	5	-2	3	-1	0	-1	0	1	0
y_2	1	1	-1	2	0	0	-1	0	1

bfs $(0, 0, 0, 4, 0, 0, 5, 1)$, basis $\{4, 7, 8\}$.

- Pick a pivot column.
- We pick 3. Column 2 would have also worked. What is the pivot row?

		x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2
$-\xi$	6	-1	2	1	0	-1	-1	0	0
x_4	4	2	-1	2	1	0	0	0	0
y_1	5	-2	3	-1	0	-1	0	1	0
y_2	1	1	-1	2	0	0	-1	0	1

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- Pick a pivot column.
- We pick 3. Column 2 would have also worked. What is the pivot row?
- Row 3.What will the new basis be?

		x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2
$-\xi$	6	-1	2	1	0	-1	-1	0	0
x_4	4	2	-1	2	1	0	0	0	0
y_1	5	-2	3	-1	0	-1	0	1	0
y_2	1	1	-1	2	0	0	-1	0	1

bfs $(0, 0, 0, 4, 0, 0, 5, 1)$, basis $\{4, 7, 8\}$.

- Pick a pivot column.
- We pick 3. Column 2 would have also worked. What is the pivot row?
- Row 3.What will the new basis be?
- $\{3, 4, 7\}$.

		x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2
$-\xi$	$11/2$	$-3/2$	$5/2$	0	0	-1	$-1/2$	0	$-1/2$
x_4	3	1	0	0	1	0	1	0	-1
y_1	$11/2$	$-3/2$	$5/2$	0	0	-1	$-1/2$	1	$1/2$
x_3	$1/2$	$1/2$	$-1/2$	1	0	0	$-1/2$	0	$1/2$

bfs $(0, 0, 1/2, 3, 0, 0, 11/2, 0)^T$, basis $\{3, 4, 7\}$.

- Pick a pivot column.

		x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2
$-\xi$	$11/2$	$-3/2$	$5/2$	0	0	-1	$-1/2$	0	$-1/2$
x_4	3	1	0	0	1	0	1	0	-1
y_1	$11/2$	$-3/2$	$5/2$	0	0	-1	$-1/2$	1	$1/2$
x_3	$1/2$	$1/2$	$-1/2$	1	0	0	$-1/2$	0	$1/2$

bfs $(0, 0, 1/2, 3, 0, 0, 11/2, 0)^T$, basis $\{3, 4, 7\}$.

- Pick a pivot column.
- Only option 2. What is the pivot row?

		x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2
$-\xi$	11/2	$-3/2$	5/2	0	0	-1	$-1/2$	0	$-1/2$
x_4	3	1	0	0	1	0	1	0	-1
y_1	11/2	$-3/2$	5/2	0	0	-1	$-1/2$	1	1/2
x_3	1/2	1/2	$-1/2$	1	0	0	$-1/2$	0	1/2

bfs $(0, 0, 1/2, 3, 0, 0, 11/2, 0)^T$, basis $\{3, 4, 7\}$.

- Pick a pivot column.
- Only option 2. What is the pivot row?
- Row 2. What will the new basis be?

		x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2
$-\xi$	$11/2$	$-3/2$	$5/2$	0	0	-1	$-1/2$	0	$-1/2$
x_4	3	1	0	0	1	0	1	0	-1
y_1	$11/2$	$-3/2$	$5/2$	0	0	-1	$-1/2$	1	$1/2$
x_3	$1/2$	$1/2$	$-1/2$	1	0	0	$-1/2$	0	$1/2$

bfs $(0, 0, 1/2, 3, 0, 0, 11/2, 0)^T$, basis $\{3, 4, 7\}$.

- Pick a pivot column.
- Only option 2. What is the pivot row?
- Row 2. What will the new basis be?
- $\{2, 3, 4\}$.

		x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2
$-\xi$	0	0	0	0	0	0	0	-1	-1
x_4	3	1	0	0	1	0	1	1	-1
x_2	$11/5$	$-3/5$	1	0	0	$-2/5$	$-1/5$	$2/5$	$1/5$
x_3	$16/10$	$2/10$	0	1	0	$-1/5$	$-6/10$	$1/5$	$6/10$

What is the bfs?

		x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2
$-\xi$	0	0	0	0	0	0	0	-1	-1
x_4	3	1	0	0	1	0	1	1	-1
x_2	$11/5$	$-3/5$	1	0	0	$-2/5$	$-1/5$	$2/5$	$1/5$
x_3	$16/10$	$2/10$	0	1	0	$-1/5$	$-6/10$	$1/5$	$6/10$

What is the bfs? bfs $(0, 11/5, 16/10, 3, 0, 0, 0, 0)^T$ basis $\{2, 3, 4\}$

		x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2
$-\xi$	0	0	0	0	0	0	0	-1	-1
x_4	3	1	0	0	1	0	1	1	-1
x_2	$11/5$	$-3/5$	1	0	0	$-2/5$	$-1/5$	$2/5$	$1/5$
x_3	$16/10$	$2/10$	0	1	0	$-1/5$	$-6/10$	$1/5$	$6/10$

What is the bfs? bfs $(0, 11/5, 16/10, 3, 0, 0, 0, 0)^T$ basis $\{2, 3, 4\}$

So $\{2, 3, 4\}$ should be a basis for the original problem. What do we do now?

		x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2
$-\xi$	0	0	0	0	0	0	0	-1	-1
x_4	3	1	0	0	1	0	1	1	-1
x_2	11/5	$-3/5$	1	0	0	$-2/5$	$-1/5$	2/5	1/5
x_3	16/10	2/10	0	1	0	$-1/5$	$-6/10$	1/5	6/10

What is the bfs? bfs $(0, 11/5, 16/10, 3, 0, 0, 0, 0)^T$ basis $\{2, 3, 4\}$

So $\{2, 3, 4\}$ should be a basis for the original problem. What do we do now? Remove columns corresponding to the auxilliary variables (i.e. the variables y_1, y_2) from the tableau.

		x_1	x_2	x_3	x_4	x_5	x_6
$-z$	0	1	-1	1	0	0	0
x_4	3	1	0	0	1	0	1
x_2	$11/5$	$-3/5$	1	0	0	$-2/5$	$-1/5$
x_3	$16/10$	$2/10$	0	1	0	$-1/5$	$-6/10$

bfs $(0, 11/5, 16/10, 3, 0, 0, 0, 0)^T$, basis $\{2, 3, 4\}$.

- What do we do now? Can we pivot?

		x_1	x_2	x_3	x_4	x_5	x_6
$-z$	0	1	-1	1	0	0	0
x_4	3	1	0	0	1	0	1
x_2	$11/5$	$-3/5$	1	0	0	$-2/5$	$-1/5$
x_3	$16/10$	$2/10$	0	1	0	$-1/5$	$-6/10$

bfs $(0, 11/5, 16/10, 3, 0, 0, 0, 0)^T$, basis $\{2, 3, 4\}$.

- What do we do now? Can we pivot?
- Remove basic variables x_2 and x_3 from Row 0

		x_1	x_2	x_3	x_4	x_5	x_6
$-z$	$3/5$	$1/5$	0	0	0	$-1/5$	$2/5$
x_4	3	1	0	0	1	0	1
x_2	$11/5$	$-3/5$	1	0	0	$-2/5$	$-1/5$
x_3	$16/10$	$2/10$	0	1	0	$-1/5$	$-6/10$

bfs $(0, 11/5, 16/10, 3, 0, 0, 0, 0)^T$, basis $\{2, 3, 4\}$.

- What is the pivot row and what is the pivot column?

		x_1	x_2	x_3	x_4	x_5	x_6
$-z$	$3/5$	$1/5$	0	0	0	$-1/5$	$2/5$
x_4	3	1	0	0	1	0	1
x_2	$11/5$	$-3/5$	1	0	0	$-2/5$	$-1/5$
x_3	$16/10$	$2/10$	0	1	0	$-1/5$	$-6/10$

bfs $(0, 11/5, 16/10, 3, 0, 0, 0, 0)^T$, basis $\{2, 3, 4\}$.

- What is the pivot row and what is the pivot column?
- Pick 6 as the pivot column and 1 is the pivot row.

		x_1	x_2	x_3	x_4	x_5	x_6
$-z$	$-3/5$	$-1/5$	0	0	$-2/5$	$-1/5$	0
x_6	3	1	0	0	1	0	1
x_2	$14/5$	$-2/5$	1	0	$1/5$	$-2/5$	0
x_3	$17/5$	$4/5$	0	1	$3/5$	$-1/5$	0

- This tableau corresponds to the basic solution $(0, 14/5, 17/5, 0, 0, 3)$ which gives $-z = 3/5$.

		x_1	x_2	x_3	x_4	x_5	x_6
$-z$	$-3/5$	$-1/5$	0	0	$-2/5$	$-1/5$	0
x_6	3	1	0	0	1	0	1
x_2	$14/5$	$-2/5$	1	0	$1/5$	$-2/5$	0
x_3	$17/5$	$4/5$	0	1	$3/5$	$-1/5$	0

- This tableau corresponds to the basic solution $(0, 14/5, 17/5, 0, 0, 3)$ which gives $-z = 3/5$.
- Is this solution optimal?

		x_1	x_2	x_3	x_4	x_5	x_6
$-z$	$-3/5$	$-1/5$	0	0	$-2/5$	$-1/5$	0
x_6	3	1	0	0	1	0	1
x_2	$14/5$	$-2/5$	1	0	$1/5$	$-2/5$	0
x_3	$17/5$	$4/5$	0	1	$3/5$	$-1/5$	0

- This tableau corresponds to the basic solution $(0, 14/5, 17/5, 0, 0, 3)$ which gives $-z = 3/5$.
- Is this solution optimal?
- Yes, because we do not have negative entries in Row 0.

		x_1	x_2	x_3	x_4	x_5	x_6
$-z$	$-3/5$	$-1/5$	0	0	$-2/5$	$-1/5$	0
x_6	3	1	0	0	1	0	1
x_2	$14/5$	$-2/5$	1	0	$1/5$	$-2/5$	0
x_3	$17/5$	$4/5$	0	1	$3/5$	$-1/5$	0

- This tableau corresponds to the basic solution $(0, 14/5, 17/5, 0, 0, 3)$ which gives $-z = 3/5$.
- Is this solution optimal?
- Yes, because we do not have negative entries in Row 0.
- The optimal value is $-3/5$ at the optimal basic feasible solution $(0, 14/5, 17/5, 0, 0, 3)^T$