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Math 482 Notes: Primal Dual simplex

We start with an LP in standard form which we call P,

$$\max c^T x \text{ such that } Ax = b, x \geq 0.$$

We assume $b \geq 0$, if $b_i < 0$, we can multiply constraint i by -1. We call the dual D

$$\min b^T y \text{ such that } A^T y \geq c.$$

Complementary slackness says the following: (a_i^T is the ith row of A and A_j is the jth column of A) if x and y are feasible for P and D, respectively, then x and y are both optimal if and only if

$$y_i(b_i - a_i^T x) = 0 \qquad \forall i \in [m] \tag{1}$$

$$(A_j^T y - c_j)x_j = 0 \qquad \forall j \in [n]. \tag{2}$$

Note that since P is in equational form, when x is feasible for P, (1) is always satisfied.

We start with some y that is feasible for D. This will always be given or **easily** obtained (e.g. $y = 0$ is feasible).

Our goal now is to write an LP that checks if y is optimal. Let

$$J = \{j \in [n] : A_j^T y = c_j\}.$$

By complementary slackness, if y is optimal, then there exists $x \in \mathbb{R}^n$ such that $Ax = b$ and $x \geq 0$ and $x_j = 0$ for all $j \notin J$. Furthermore, any such x is optimal for P.

So we define the following LP which we call the *restricted primal* (RP): Let $\hat{x} = \left[\begin{array}{c} x^r \\ \hline x_J \end{array}\right]$ where $x^r \in \mathbb{R}^m$ are new variables

$$\text{Max } \xi = [-\mathbf{1}^T|\mathbf{0}]\hat{x} \text{ subject to } [I_m|A_J]\,\hat{x} \text{ and } \hat{x} \geq 0.$$

Note that (RP) is feasible ($x^r = b$ and $x_J = 0$) and bounded above by 0, so we can solve (RP) with simplex (we usually use revised simplex), and get a optimal solution. Let $A = [a_{i,j}]$ be the final tableau. If $\xi_{opt} = -a_{0,0} = 0$, then $x^r = 0$, so $A_J x_J = b$, and if we let $x_j = 0$ for all $j \notin J$, then $Ax = b$ and $x \geq 0$. Therefore, x and y are optimal for P and D, respectively.

So assume $\xi_{opt} < 0$. This implies that y was not optimal. We wish to improve y using an optimal solution $\bar{y}$ to the *dual of the restricted primal* (DRP). Using the dualization recipe we have that (DRP) is

$$\min b^T y \text{ subject to } A_J^T y \geq 0 \text{ and } y_i \geq -1 \text{ for all } i \in [m].$$

We can extract an optimal solution to (DRP) from the final tableau A, since when B is a basis that correspond to an optimal solution of (RP), $c_B A_B^{-1}$ is an optimal solution to (DRP). So we set $\overline{y} = c_B A_B^{-1}$ and note that, for any $j \in [n]$, $a_{0,j} = c_j - \overline{y}^T A_j$. So since, for any $j \in [m]$, $c_j = -1$ and $A_j = e_j$ (here e_j is the jth standard basis vector) we have that $a_{0,j} = -1 - \overline{y}_j$. So we have that for any $i \in [m]$, $\overline{y}_i = -1 - a_{0,i}$.

Note that, by strong duality, $b^T\overline{y} = \xi_{opt}$ and recall that we are assuming $\xi_{opt} < 0$. Let $y^* := y + \theta\overline{y}$ for some $\theta > 0$. We will pick θ so that y^* is a new, better feasible solution for D. First, note that

$$b^T y^* = b^T y + \theta b^T \overline{y} < b^T y$$

so y^* will indeed be a better feasible solution than y if it is feasible and $\theta > 0$.

If $A_j^T\overline{y} \geq 0$ for all $j \notin J$, then, since $\overline{y}$ is feasible for (DRP), $A_j^T\overline{y} \geq 0$ for all $j \in [n]$, and $A_j^T y^* = A_j^T y + \theta A_j^T \overline{y} \leq c_j$ for any $\theta > 0$. So as we let θ go to infinity, y^* is always feasible for D and $b^T y^*$ goes to $-\infty$. So D is unbounded and P is infeasible.

Otherwise, we let

$$\theta = \min_{\substack{j \notin J \\ A_j^T\overline{y} < 0}} \left\{ \frac{c_j - A_j^T y}{A_j^T \overline{y}} \right\}$$

Note that $\theta > 0$. Furthermore, for any $j \in [n]$, either $A_j^T\overline{y} \geq 0$ and $A_j^T y^* \geq c_j$ for any $\theta > 0$, or $j \notin J$ and $A_j^T\overline{y} < 0$, so

$$A_j^T y^* = A_j^T y + \theta A_j^T \overline{y} \geq c_j + \frac{c_j - A_j^T y}{A_j^T \overline{y}} A_j^T \overline{y} = c_j.$$

Therefore, y^* is feasible and $b^T y^* < b^T y$. We can now replace y with y^* and repeat the process, i.e. construct a new (RP), solve it with simplex, etc. We are finished when the optimal value of (RP) is 0.