08:14 Wednesday 8th April, 2015

Math 482 Notes: Primal Dual min-cost path

Let $G = (V, E)$ be a directed graph with cost function $c : E \to \mathbb{R}^+$ on the edges. We let $m := |E|$ be the number of edges of G and $n := |V|$ be the number of vertices of G. We wish to find a shortest or min-cost path from vertex s to vertex t. We let A be the incidence matrix for G with the row corresponding to t removed and we attempt to find a solution to the following LP:

$$\text{(P) } \min c^T f \text{ such that } Af = d, f \geq 0.$$

Here d is defined by

$$d_i = \begin{cases} 1 & \text{if } i = s \\ 0 & \text{otherwise} \end{cases},$$

so we essentially want one unit of flow to leave s, and flow to be conversed at every other vertex except t. This implies that one unit of flow will enter t. We can remove the last row (or any row) of A, because A does not have full rank since the sum of the rows of an incident matrix of any directed graph is 0.

The dual is

$$\text{(D) } \max \pi_s \text{ such that } \pi_i - \pi_j = c_{ij} \forall (i, j) \in E, \pi_t = 0.$$

Notice how we have associated the variables in the dual with the vertices in of G. Here we add the extra variable $\pi_t = 0$ and set it to 0, this is necessary because there is not a row corresponding to t in the original primal LP.

Exercise: If π is feasible for (D), prove that for any $v \in V$, π_v is at most the cost of a shortest path from v to t.

To start the primal dual process, assume the π is feasible for D, ($\pi = 0$ is feasible if $c \geq 0$) and let $J = \{(i, j) \in E : \pi_i - \pi_j = c_{ij}\}$. Call a path in G a *J-path* if it consists entirely of edges from the set J. Let

$$W = \{v \in V : \text{ there is a } J\text{-path from } x \text{ to } t \text{ in } G\}$$

and let $\overline{W} = V \setminus W$.

Exercise: If π is feasible for (D), prove that for any $v \in W$, π_v is exactly the cost of a shortest path from v to t.

Now we can write the restricted primal program, let $x_r \in \mathbb{R}^m$ and $\hat{f} = [x^r | f_J]^T$

$$\text{(RP) } \min \xi = [\mathbf{1}^T | \mathbf{0}^T] \hat{f} \text{ such that } [I_m | A_J] \hat{f} = d, \hat{f} \geq 0.$$

Clearly (RP) is feasible and bounded, so we can let ξ_{opt} be the optimal value of (RP). The following is the dual of the restricted primal

$$\text{(DRP)} \quad \text{such that} \quad \begin{array}{lll} \max \overline{\pi}_s & & \\ \overline{\pi}_i - \overline{\pi}_j & \leq 0 & \text{for all } (i, j) \in J \\ \overline{\pi}_i - \overline{\pi}_j & \leq 1 & \text{for all } (i, j) \in E \\ \overline{\pi}_t & = 0. & \end{array}$$

Assume that $\overline{\pi}$ is feasible for (DRP). Note that if $\overline{\pi}$ is optimal and $\overline{\pi}_s = 0$, then $\xi_{opt} = 0$ and we are done. Also, note that if $\overline{\pi}$ is feasible for (DRP) and $\overline{\pi}_s = 1$, then $\overline{\pi}$ is clearly optimal.

Let $x \in W$. Since $x \in W$, there exists a J-path $x = v_1 v_w \cdots v_{d-1} v_d = t$ from x to t. Since $\overline{\pi}$ is feasible we have that inequalities

$$0 = \overline{\pi}_t = \overline{\pi}_{v_d} \geq \overline{\pi}_{v_{d-1}} \geq \cdots \geq \overline{\pi}_{v_1} = \overline{\pi}_x,$$

so $\overline{\pi}_x <= 0$. Since $\overline{\pi} = \mathbf{0}$ is feasible for (DRP), when $s \in W$, $\overline{\pi} = \mathbf{0}$ is optimal and we are done.

Let

$$\overline{\pi}_i = \begin{cases} 0 & \text{if } i \in W \\ 1 & \text{if } i \in \overline{W} \end{cases}.$$

We claim that $\overline{\pi}$ is feasible for DRP. To see this note that for every $(i,j) \in E$, $\overline{\pi}_i - \overline{\pi}_j \leq 0$, unless $i \in \overline{W}$ and $j \in W$ and $\overline{\pi}_i - \overline{\pi}_j = 1$, but in this case, by the definition of W, $(i,j) \notin J$. We compute

$$\theta = \min_{\substack{(i,j) \in J \\ \overline{\pi}_i - \overline{\pi}_j > 0}} \left\{ \frac{c_{ij} - (\pi_i - \pi_j)}{\overline{\pi}_i - \overline{\pi}_j} \right\} = \min_{\substack{(i,j) \in E \\ i \in \overline{W} \text{ and } j \in W}} \{ c_{ij} - (\pi_i - \pi_j) \}$$

So we now have our the following algorithm:

(1) Start with $\pi = 0$ and let $W = \{t\}$ and $\overline{W} = V \setminus \{t\}$.
(2) Compute $\theta = \min\limits_{\substack{(i,j) \in E \\ i \in \overline{W} \text{ and } j \in W}} \{ c_{ij} - (\pi_i - \pi_j) \}$.
(3) Add θ to π_i if $i \in \overline{W}$.
(4) Add i to W and remove it from $\overline{W}$, if there exists an edge (i,j) such that $j \in \overline{W}$ and $c_{ij} - (\pi_i - \pi_j) = \theta$. Note that there will always exist at least one such vertex i.
(5) If $s \in W$ we are done, π_s is the length of the shortest path from s to t. Otherwise, repeat from step 2.

Exercise: Prove that once $x \in W$ it is in W on every iteration.