08:36 Wednesday 8$^{\text{th}}$ April, 2015

Math 482 Notes: Primal Dual and max flow

Let $G = (V, E)$ be a directed graph with capacity function $b : E \to \mathbb{R}^+$. We let $m := |E|$ be the number of edges of G and $n := |V|$ be the number of vertices of G. A flow from vertex s to vertex t is a function $f \in E \to \mathbb{R}^+$ such that for every $i \in V \setminus \{s, t\}$

$$\sum_{(i,j)\in E} f_{ij} - \sum_{(j,i)\in E} f_{ji} = 0.$$

The *value* of a flow is

$$|f| := \sum_{(s,j)\in E} f_{sj} - \sum_{(j,s)\in E} f_{js}$$

Note that we must have

$$-|f| = \sum_{(t,j)\in E} f_{tj} - \sum_{(j,t)\in E} f_{jt}$$

since

$$\sum_{i\in V}\left[\sum_{(i,j)\in E} f_{ij} - \sum_{(j,i)\in E} f_{ji}\right] = \sum_{(i,j)\in E} f_{ij} - \sum_{(i,j)\in E} f_{ij} = 0.$$

We wish to find the max-flow from a vertex s to a vertex t. Let A be the incidence matrix for G and

$$\text{(MF) max } v \text{ such that } Af + dv = 0, f \leq b, f \geq 0.$$

Here d is slightly different from the min cost path problem

$$d_i = \begin{cases} -1 & \text{if } i = s \\ 1 & \text{if } i = s \\ 0 & \text{otherwise} \end{cases}.$$

Exercise: Show that for any f and v feasible for MF, f is an s, t flow and the value of f is v.

We now want to use the primal dual method. It is convenient in this case to treat our original problem as the dual to a **minimization** linear program in standard form. Note that $Af + d = 0$ if and only if $Af + d \leq 0$ because for any $j \in [m]$, $\sum_{i\in n} a_{ij} = 0$ and $\sum_{i\in[n]} d_i = 0$ (recall that the number of columns in A is the number of edges in G and the number of rows in G is the number of vertices in G).

So we can write the LP as

$$\text{(MF') max } v \text{ such that } Af + dv \leq 0, f \leq b, -f \leq 0.$$

We know do not write the actual corresponding primal or restricted primal problems. We only observe that when the dual is of the form

$$\max c^T x \text{ such that } Ax \leq b,$$

as our problem is, the dual of the restricted primal is the same as the dual except that

- all of the variables are bounded above by 1,
- only the constraints that hold without slack in the dual are in the dual of the restricted primal,

- the constraints that are transferred from the dual to the dual of the restricted primal are the same except the right hand side is 0.

So if f and v are feasible for (MF') the dual of the restricted primal is

$$\begin{array}{lll} & \max \overline{v} & \\ \text{such that} & A\overline{f} + d\overline{v} & \leq 0, \\ & \overline{f}_{ij} & \leq 0 \text{ for all } (i,j) \in E \text{ such that } f_{ij} = b_{ij} \\ & -\overline{f}_{i,j} & \leq 0 \text{ for all } (i,j) \in E \text{ such that } -f_{ij} = 0 \\ & \overline{f}_{i,j} & \leq 1 \text{ for all } (i,j) \in E \\ & \overline{v} & \leq 1 \end{array}$$

Recall that $a_i^T f + d_i v = 0$ for every $i \in V$, so every constraint $a_i^T \overline{f} + d_i \overline{v} = 0$ is active in the dual of the restricted primal, i.e. we have the n constraints given by $A^T \overline{f} + d\overline{v} = 0$ in the dual of the restricted primal. Note that the dual restricted primal also asks for a flow $\overline{f}$, except that $\overline{f}_i j$ can be negative. The constraint on edge $(i,j) \in E$ is 0 if $f_{ij} = b_i j$ and 1 otherwise. We also have the $\overline{f}_{ij}$ must be positive if $f_{ij} = 0$.

Let $\overline{f}$, $\overline{v}$ be an optimal solution to the dual of the restricted primal. We have that $0 \leq \overline{v} \leq 1$. This follows from the fact that $\overline{v} \leq 1$ by the last constraint and a zero flow is feasible. Furthermore, by the primal dual method, if $\overline{f} = 0$, it must be that f is an optimal solution to (MF') and we are done.

So our goal now is to find an optimal solution to the dual. So we make the following definition, a path P is an f-*augmenting path* if whenever (i,j) is an edge on P either

- $(i,j) \in E$ and $f_{ij} < b_{ij}$ (these are the *forward arcs* and they correspond to $\overline{f}_{ij} = 1$)
- $(j,i) \in E$ and $f_{ij} > 0$ (these are the *backward arcs* and they correspond to $\overline{f}_{ij} = -1$)

Note here that if (i,j) is an edge in P that it does not necessarily have to be an edge in G, but if it is not an edge in G, then (j,i) must be an edge in G. If P is an f-augmenting path from s to t, and, for all other edges $(i,j) \in E$, we set $\overline{f}_{ij} = 0$, then $\overline{f}$ and $\overline{v} = 1$ together from an optimal solution to the dual of the restricted primal. You can assume that if no such path exists, then the optimal value of the dual of the restricted primal is 0, which implies that f is optimal.

We now need to use $\overline{f}$ to find a feasible solution f^* to (MF') that is better than f for (MF'). We set $f^* = f + \theta\overline{f}$ and pick θ as large as possible so that f^* is still feasible for the (MF'). It is not hard to see that the proper θ value is

$$\min\{ \min_{\substack{(i,j) \\ \text{a forward arc on } P}} \{b_{ij} - f_{ij}\}, \min_{\substack{(j,i) \\ \text{a backward arc on } P}} \{f_{ij}\}, \}$$

The primal dual algorithm has then given us the following algorithm originally due to Ford and Fulkerson.

(1) Init $f = 0$ and $v = 0$.
(2) Find P an f augmenting s,t-path and let $\overline{f}$ be the corresponding solution to the dual of the restricted primal
(3) If no such path exists f is optimal.
(4) Otherwise, add $\theta\overline{f}$ to f and repeat from step 2.