16:45 Tuesday 14th April, 2015

Math 482 Notes: Max flow and revised simplex

Let $G = (V, E)$ be a directed graph with distinct vertices $s, t \in V$. Let $P_1, \dots, P_p$ be all of the paths from s to t in G. We define the matrix $D = \{d_{ij}\}$ by

$$d_{ij} = \begin{cases} 1 & \text{if edge } e_i \text{ is on the path } P_j \\ 0 & \text{otherwise.} \end{cases}$$

We let b_i be the capacity on edge e_i, then the max flow LP is

$$\begin{array}{lrl} \max & \sum_{j=1}^{p} f_j & \\ \text{subject to} & Df & \le b \\ & f & \ge 0 \end{array}$$

Where $Df \le b$ is just shorthand for $\sum_{e_i \in E} \sum_{j=1}^{p} d_{ij} f_j \le b_i$ and $f \ge 0$ is shorthand for $f_j \ge 0$ for all $j \in \{1, \dots, p\}$. We add slack variables $s \in R^m$ to put the LP in standard form. In detail, let $\hat{c} = [\mathbf{0^T} | \mathbf{1^T}]$, $\hat{f} = \left[\frac{s}{f}\right]$, and $\hat{D} = [I_m | D]$, so the LP in standard form is then:

$$\max \hat{c}^T \hat{f} \text{ s.t } \hat{D}\hat{f} \le b, \hat{f} \ge 0.$$

Our CARRY-0 matrix is then,

$$\begin{array}{c|c|c|} & 0 & \mathbf{0^T} \\ \hline s_1 & & \\ \vdots & b & I_m \\ s_m & & \\ \hline \end{array}$$

and at step ℓ we have CARRY-ℓ

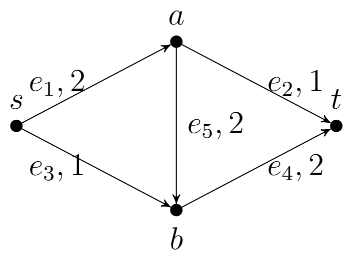
$$\begin{array}{c|c|c|} & -|f| & -\pi^T \\ \hline \hat{f}_{j_1} & & \\ \vdots & A_B^{-1} b & A_B^{-1} \\ \hat{f}_{j_m} & & \\ \hline \end{array}$$

Here $|f|$ is the current value of the flow, and $B = \{j_1, \dots, j_m\}$ is the current basis, (row i is solve for element $\hat{f}_i$ and $\hat{f}_i$ either corresponds to a slack variable or one of the s, t-paths $P_1, \dots, P_p$).

To pivot, we need $j \in [p]$ such that $\bar{c_j} = C_j - \pi^T D_j > 0$, since $c_j = 1$ for every $j \in P$, this is equivalent to $\pi^T D_j < 1$. Note that if some $\pi_i < 0$ for some $i \in [m]$, then it is profitable to bring the slack variable s_i into the basis, i.e. we can pivot on row i of the CARRY matrix. We can therefore assume that $\pi \ge 0$.

Now we can view π_i as a weight/cost on edge i in the graph and find a shortest path from s, t subject to the weight/cost function π_i. If the cost of this path is less than 1, than it is profitable to bring f_i into the basis.

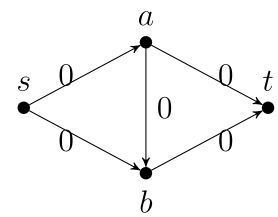
The following is an example. We start with the following directed graph (the capacities of the edges are listed after the edge labels):



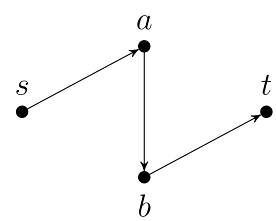
This is CARRY-0 with an additional column which will be explained later.

	0	0	0	0	0	0	1
s_1	2	1	0	0	0	0	(1)
s_2	1	0	1	0	0	0	0
s_3	1	0	0	1	0	0	0
s_4	2	0	0	0	1	0	1
s_5	2	0	0	0	0	1	1

We find the shortest s, t-path in the following graph since $-\pi_T = [0, 0, 0, 0, 0]$



Every path has cost 0, so any path will work. We pick the path

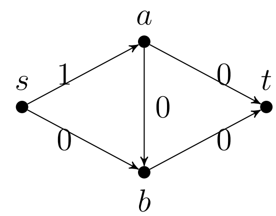


We call this path *sabt* and, since it contains the edges e_1, e_4 and e_5, it corresponds to the column $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ in D. We also have that the relative cost of the column is $\bar{c}_j = c_j - \pi^T \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = 1$ (recall that $c_j = 1$ for every path). We now can see how the additional column was constructed. It consists of $A_B^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ in rows 1 through 5 and $\bar{c}_j = 1$ in row 0. We pivot, as we would in normal simplex, on the circled entry.

This is CARRY-1 with the pivot column (note that we do not know what the pivot column is until we complete the steps below, it is just listed now for convenience):

	-2	-1	0	0	0	0	1
sabt	2	1	0	0	0	0	0
s_2	1	0	1	0	0	0	0
s_3	1	0	0	1	0	0	1
s_4	0	-1	0	0	1	0	(1)
s_5	0	-1	0	0	0	1	0

We find the shortest s,t-path in the following graph since $-\pi_T = [-1, 0, 0, 0, 0]$

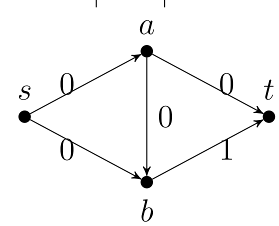


The shortest path is sbt and it has cost $0 < 1$, so we bring the path into the basis. We compute $A_B^{-1}\begin{bmatrix}0\\0\\1\\1\\0\end{bmatrix}$ and $\bar{c}_j = 1 - \pi^T\begin{bmatrix}0\\0\\1\\1\\0\end{bmatrix} = 1$. We pivot on the circled entry.

This is CARRY-2 with an pivot column is:

	-2	0	0	0	-1	0	1
$sabt$	2	1	0	0	0	0	1
s_2	1	0	1	0	0	0	(1)
s_3	1	1	0	1	-1	0	1
sbt	0	-1	0	0	1	0	-1
s_5	0	-1	0	0	0	1	-1

We find the shortest s,t-path in the following graph since $-\pi_T = [0, 0, 0, -1, 0]$



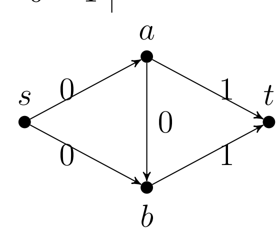
The shortest path is sat. We compute $A_B^{-1}\begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}1\\1\\1\\-1\\-1\end{bmatrix}$ and $\bar{c}_j = 1 - \pi^T\begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix} = 1$.

We pivot on the circled entry.

This is CARRY-3 with an pivot column is:

	-3	0	-1	0	-1	0
$sabt$	1	1	-1	0	0	0
sat	1	0	1	0	0	0
s_3	0	1	-1	1	-1	0
sbt	1	-1	1	0	1	0
s_5	1	-1	1	0	0	1

We find the shortest s,t-path in the following graph since $-\pi_T = [0, -1, 0, -1, 0]$



Paths sat, sbt and $sabt$ all have cost 1, so no path has cost less than 1. This means we are finished and the max flow is 3. This is a flow of 1 on path $sabt$ and flow of 1 on sat and a flow of 1 on sbt. Note that we also have one unit of slack to e_5 and no slack anywhere else. We can see this by observing that $s_5 = 1$ in the final basic feasible solution.