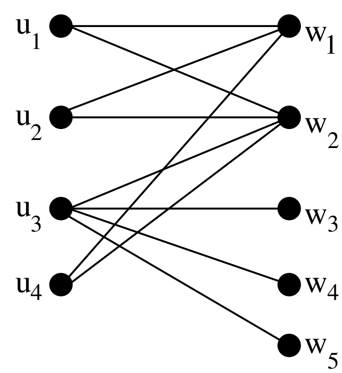
Math 482 **HW9** **4/13/2014** Name: ______________________________

Due Friday, April 24, 2014

1. Using maximum flows, find a maximum matching in the bipartite graph below on the left. Prove that the matching is optimal. (You do not need to show how you found the maximum flow).



2. Assume that the following modified version Farkas lemma is true:
For any $A \in Q^{m\times n}$ and $b \in \mathbb{Q}^n$, there exists $\omega \in \mathbb{N}$ and $y \in \mathbb{Q}^m$ exactly one of the following two statements hold:

   - there exists $x \in \mathbb{Q}^n$ such that $Ax \leq b$, or
   - there exists $y \in \mathbb{Q}^m$ such that $y^T A = 0$, $y \geq 0$, $y^T b = -1$ and $|y_i| < 2^\omega$ for every $i \in [m]$.

   Let $A \in \mathbb{Q}^{m\times n}$ and $b \in \mathbb{Q}^n$ and $\omega \in \mathbb{N}$ be as in the modified version of Farkas Lemma above. Show that if $\eta = 1/(2^\omega m)$ and there exists $\tilde{x} \in \mathbb{Q}^n$ then $A\tilde{x} \leq b + \eta\mathbf{1}$, then there exists a solution to $Ax \leq b$.

3. Consider the following integer program P:

$$\begin{array}{lrrcl} \text{minimize} & z = x_1 & & & \\ \text{subject to} & 3x_1 & -100x_2 & \geq & 1 \\ & 3x_1 & -101x_2 & \leq & 1 \\ & x_1, & x_2 & \geq & 0 \\ & x_1 \in \mathbb{Z}, & x_2 \in Z & & \end{array}$$

   Solve the linear programming relaxation of P, obtaining an optimal solution x^* with cost z^* (You can solve the linear programming relaxation in manner you wish.) Obtain an integer vector x from x^* by rounding each component to the nearest integer. Is x an optimal solution to the integer program P? If it is not, find an optimal solution to the integer program P. (You can solve the integer programming in any manner you want.)

4. Suppose you are playing a game in which you have a set of tiles, each containing one of the letters $l_1, \ldots, l_r$. You are to construct a list of valid words using your assigned tiles. The valid words are denoted $w_1, \ldots, w_s$ and you can construct each word at most once. Let

   - $c_{i,j}$ be the number of times letter l_i appears in word w_j,
   - d_i be the number of tiles containing letter l_i that you have, and
   - b_j be the number of points you receive for constructing word w_j.

   Construct a integer linear program to determine an optimal set of words to construct.

5. Suppose that in the game described in the previous question you receive a bonus of $a_{i,j}$ if the pair of words w_i and w_j are both constructed. For example, if the words w_1, w_2 and w_3 are constructed, then the score is $b_1 + b_2 + b_3 + a_{1,2} + a_{1,3} + a_{2,3}$. Construct a integer linear program to determine an optimal set of words to construct for this modified game. (Make sure that your program is still linear.)