An example of the primal–dual simplex method

Suppose we are given the problem **P**:

$$\text{Maximize } z = -x_1 - 3x_2 - 3x_3 - x_4$$

$$\text{subject to} \quad \begin{cases} 3x_1 & +4x_2 & -3x_3 & +x_4 & = & 2, \\ 3x_1 & -2x_2 & +6x_3 & -x_4 & = & 1, \\ 6x_1 & +4x_2 & & +x_4 & = & 4, \\ x_1, & x_2, & x_3, & x_4 & \geq & 0. \end{cases} \tag{1}$$

The dual to **P** is of course the following **D**

$$\text{Minimize } w = 2\pi_1 + \pi_2 + 4\pi_3$$

$$\text{subject to} \quad \begin{cases} 3\pi_1 & +3\pi_2 & +6\pi_3 & \geq & -1, \\ 4\pi_1 & -2\pi_2 & +4\pi_3 & \geq & -3, \\ -3\pi_1 & +6\pi_2 & & \geq & -3, \\ \pi_1 & -\pi_2 & +\pi_3 & \geq & -1. \end{cases} \tag{2}$$

Somebody tells us that probably vector $\pi = (-1/3,\, 0,\, 0)^T$ is an optimal vector in **D**. Note that the value of w with this π is $-2/3$. We start checking this version using complementary slackness. First, we plug this vector in **D** and see that it is a feasible vector and only the first inequality is binding. Hence our first set J is $\{1\}$. In particular, if π is an optimal vector in **D**, then in the corresponding optimal vector $\mathbf{x}$ of **P** only coordinate x_1 can be non-zero. We try to find it by solving the following *restricted primal problem* **RP1**:

$$\text{Maximize } \xi = -x_1^r - x_2^r - x_3^r$$

$$\text{subject to} \quad \begin{cases} 3x_1 & +x_1^r & & & = & 2, \\ 3x_1 & & +x_2^r & & = & 1, \\ 6x_1 & & & +x_3^r & = & 4, \\ x_1, & x_1^r, & x_2^r, & x_3^r & \geq & 0. \end{cases} \tag{3}$$

Normally, we would use the revised simplex to solve it. But here we will write down all the tableaus. So, the initial tableau is

		x_1	x_1^r	x_2^r	x_3^r
$y_0 = -\xi$	0	0	−1	−1	−1
x_1^r	2	3	1	0	0
x_2^r	1	3	0	1	0
x_3^r	4	6	0	0	1

Excluding x_1^r, x_2^r, and x_3^r from Row 0, we have

		x_1	x_1^r	x_2^r	x_3^r
$y_0 = -\xi$	7	12	0	0	0
x_1^r	2	3	1	0	0
x_2^r	1	3	0	1	0
x_3^r	4	6	0	0	1

We pivot on $a_{2,1}$ and get

		x_1	x_1^r	x_2^r	x_3^r
$y_0 = -\xi$	3	0	0	-4	0
x_1^r	1	0	1	-1	0
x_1	1/3	1	0	1/3	0
x_3^r	2	0	0	-2	1

This is the final tableau which proves that our $\pi = (-1/3,\, 0,\, 0)^T$ is NOT optimal. But this is not only a negative outcome, since we now know how to improve the π. Our new π^* will have the form

$$\pi^* = \pi + \theta\overline{\pi}, \tag{4}$$

where θ is a positive factor that we will find below and $\overline{\pi}$ is an optimal vector in the dual **DRP1** to **RP1** which (by definition) is as follows:

$$\text{Minimize } w^r = 2\overline{\pi}_1 + \overline{\pi}_2 + 4\overline{\pi}_3$$

subject to

$$\begin{cases} 3\overline{\pi}_1 & +3\overline{\pi}_2 & +6\overline{\pi}_3 & \geq & 0, \\ \overline{\pi}_1 & & & \geq & -1, \\ & \overline{\pi}_2 & & \geq & -1, \\ & & \overline{\pi}_3 & \geq & -1. \end{cases}$$

We can find $\overline{\pi}$, from the last tableau for **RP1**, where the vector $(0, -4, 0)$ in Row 0 is in fact $(-1, -1, -1) - (\overline{\pi}_1, \overline{\pi}_2, \overline{\pi}_3)$. Hence $(\overline{\pi}_1, \overline{\pi}_2, \overline{\pi}_3) = (-1, -1, -1) - (0, -4, 0) = (-1, 3, -1)$. Now we choose the maximum θ such that the vector $(\pi^*)^T = (-1/3, 0, 0) + \theta(-1, 3, -1)$ is feasible in **D**. Plugging this π^* into the first inequality of **D** we get the inequality

$$(\pi^*)^T \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix} = (-1/3, 0, 0) \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix} + \theta(-1, 3, -1) \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix} = -1 + \theta \cdot 0 = -1 \geq -1,$$

which holds for every θ. Similarly, plugging π^* into the second inequality of **D** we get the inequality $-4/3 - \theta 14 \geq -3$ which holds for $\theta \leq 5/42$. Plugging π^* into the third inequality of **D** we get $1 + \theta(21) \geq -3$ which holds for every positive θ. Finally, plugging π^* into the fourth inequality of **D** we get $-1/3 + \theta \cdot -5 \geq -1$ which holds for $\theta \leq 2/15$. Thus we choose $\theta = 5/42$ and hence our new $\pi = \pi*$ is $(-1/3, 0, 0)^T + \frac{5}{42}(-1, 3, -1)^T = (-\frac{19}{42}, \frac{5}{14}, -\frac{5}{42})^T$. Note that now $w = 2\frac{-19}{42} + \frac{-5}{14} + 4\frac{-5}{42} = -\frac{43}{42}$.

So, we start our cycle again. We hope that the new π is optimal. Plugging it in **D** we see that now $J = \{1, 2\}$. Thus, our new restricted primal **RP2** is

$$\text{Maximize } \xi = -x_1^r - x_2^r - x_3^r$$

subject to

$$\begin{cases} 3x_1 & +4x_2 & +x_1^r & & & = & 2, \\ 3x_1 & -2x_2 & & +x_2^r & & = & 1, \\ 6x_1 & +4x_2 & & & +x_3^r & = & 4, \\ x_1, & x_2, & x_1^r, & x_2^r, & x_3^r & \geq & 0. \end{cases} \tag{5}$$

But we do not start from scratch. We use the last tableau of the previous iteration adding there the values of the x_2-column obtained from knowing A_B^{-1}:

		x_1	x_2	x_1^r	x_2^r	x_3^r
$y_0 = -\xi$	3	0	14	0	-4	0
x_1^r	1	0	6	1	-1	0
x_1	1/3	1	$-2/3$	0	1/3	0
x_3^r	2	0	8	0	-2	1

Here, the second column was obtained using the formulas $\widetilde{c}_2 = c_2-(-1,\ 3,\ -1)\begin{pmatrix} 4 \\ -2 \\ 4 \end{pmatrix} = 0 + 14 = 14$, and $\widetilde{A}_2 = A_B^{-1}A_2 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1/3 & 0 \\ 0 & -2 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ -2/3 \\ 8 \end{pmatrix}$. Note that A_B^{-1} is in the last three rows and columns of the previous tableau.

We pivot on $a_{1,2}$ and get

		x_1	x_2	x_1^r	x_2^r	x_3^r
$y_0 = -\xi$	2/3	0	0	$-7/3$	$-5/3$	0
x_2	1/6	0	1	1/6	$-1/6$	0
x_1	4/9	1	0	1/9	2/9	0
x_3^r	2/3	0	0	$-4/3$	$-2/3$	1

This is the final tableau which proves that our new π again is not optimal. So, we again correct it using (4). Recall that our restricted dual **DRP2** is

$$\text{Minimize } w^r = 2\overline{\pi}_1 + \overline{\pi}_2 + 4\overline{\pi}_3$$

$$\text{subject to} \quad \begin{cases} 3\overline{\pi}_1 & +3\overline{\pi}_2 & +6\overline{\pi}_3 & \geq & 0, \\ 4\overline{\pi}_1 & -2\overline{\pi}_2 & +4\overline{\pi}_3 & \geq & 0, \\ \overline{\pi}_1 & & & \geq & -1, \\ & \overline{\pi}_2 & & \geq & -1, \\ & & \overline{\pi}_3 & \geq & -1. \end{cases}$$

Similarly to the previous iteration, we have $(\overline{\pi}_1, \overline{\pi}_2, \overline{\pi}_3) = (-1,-1,-1)-(-7/3,-5/3,0) = (4/3, 2/3, -1)$. To find the maximum θ such that the vector $\pi^* = (-\frac{19}{42}, \frac{5}{14}, -\frac{5}{42})^T + \theta(4/3, 2/3, -1)^T$ is feasible in **D**, we plug this π^* into all inequalities of **D**. From the first inequality we get

$$(\pi^*)^T\begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix} = (-\frac{19}{42}, \frac{5}{14}, -\frac{5}{42})\begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix} + \theta(4/3, 2/3, -1)\begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix} = -1+\theta\cdot(4+2-6) = -1 \geq -1,$$

which holds for every θ. Similarly, from the second inequality of **D** we get $-3 + \theta(16/3 - 4/3 - 4) \geq -3$ which also holds for every θ. From the third inequality of **D** we get $7/2 + \theta(-4 + 4 + 0) \geq -3$ which holds for every θ. Finally, from the fourth inequality of **D** we get $-13/14 + \theta(4/3 - 2/3 - 1) \geq -1$ which holds for $\theta \leq 3/14$.

Thus we choose $\theta = 3/14$ and hence our new π is $(-\frac{19}{42}, \frac{5}{14}, -\frac{5}{42})^T + \frac{3}{14}(4/3, 2/3, -1)^T = (-\frac{1}{6}, \frac{1}{2}, -\frac{1}{3})^T$. Note that now $w = 2\frac{-1}{6} + \frac{1}{2} + 4\frac{-1}{3} = \frac{-7}{6}$.

We start our cycle again. Now $J = \{1, 2, 4\}$. Thus, our new restricted primal **RP3** is

$$\text{Maximize } \xi = x_1^r + x_2^r + x_3^r$$

subject to

$$\begin{cases} 3x_1 & +4x_2 & +x_4 & +x_1^r & & & = & 2, \\ 3x_1 & -2x_2 & -x_4 & & +x_2^r & & = & 1, \\ 6x_1 & +4x_2 & +x_4 & & & +x_3^r & = & 4, \\ x_1, & x_2, & x_4, & x_1^r, & x_2^r, & x_3^r & \geq & 0. \end{cases}$$

We use the modified last tableau

		x_1	x_2	x_4	x_1^r	x_2^r	x_3^r
$y_0 = -\xi$	2/3	0	0	1/3	−7/3	−5/3	0
x_2	1/6	0	1	1/3	1/6	−1/6	0
x_1	4/9	1	0	−1/9	1/9	2/9	0
x_3^r	2/3	0	0	1/3	−4/3	−2/3	1

where the third column was obtained using the formulas

$$\widetilde{c}_4 = c_4 - (4/3, 2/3, -1) \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0 + 1/3 = 1/3,$$

and $\widetilde{A}_4 = A_B^{-1} A_4 = \begin{pmatrix} 1/6 & -1/6 & 0 \\ 1/9 & 2/9 & 0 \\ -4/3 & -2/3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ -1/9 \\ 1/3 \end{pmatrix}$.

We pivot on x_4-column and Row 1. The result is

		x_1	x_2	x_4	x_1^r	x_2^r	x_3^r
$y_0 = -\xi$	1/2	0	−1	0	−5/2	−3/2	0
x_4	1/2	0	3	1	1/2	−1/2	0
x_1	1/2	1	1/3	0	1/6	1/6	0
x_3^r	1/2	0	−1	0	−3/2	−1/2	1

As above, the optimal vector of the new restricted dual **DRP3** is $(\overline{\pi}_1, \overline{\pi}_2, \overline{\pi}_3)^T = (-1, -1, -1)^T - (-5/2, -3/2, 0)^T = (3/2, 1/2, -1)^T$. To find the maximum θ such that the vector $\pi^* = (-\frac{1}{6}, \frac{1}{2}, -\frac{1}{3})^T + \theta(3/2, 1/2, -1)^T$ is feasible in **D**, we do not need to check inequalities in **D** corresponding to x_1 and x_4, since they are in the basis of **RP3**. From the remaining two inequalities we get

$$(\pi^*)^T \begin{pmatrix} -3 \\ 6 \\ 0 \end{pmatrix} = (-\frac{1}{6}, \frac{1}{2}, -\frac{1}{3}) \begin{pmatrix} -3 \\ 6 \\ 0 \end{pmatrix} + \theta(3/2, -1/2, -1) \begin{pmatrix} 3 \\ -6 \\ 0 \end{pmatrix} = 7/2 + \theta \cdot (-9/2 + 3 + 0) \geq -3,$$

which holds for $\theta \leq 13/3$, and $-3+\theta(6-1-4) \geq -3$ which holds for each positive θ.

Thus we choose $\theta = 13/3$ and hence our new π is $(-\frac{1}{6}, \frac{1}{2}, -\frac{1}{3})^T + \frac{13}{3}(3/2, 1/2, -1)^T = (\frac{19}{3}, \frac{8}{3}, -\frac{14}{3})^T$. Note that now $w = -2\frac{-19}{3} + \frac{8}{3} - 4\frac{14}{3} = -\frac{10}{3}$.

We start our cycle again. Now $J = \{1, 3, 4\}$. Note that 2 is not in J anymore. The tableau corresponding to the new restricted primal **RP4** is

		x_1	x_3	x_4	x_1^r	x_2^r	x_3^r
$y_0 = -\xi$	1/2	0	3/2	0	$-5/2$	$-3/2$	0
x_4	1/2	0	$-9/2$	1	1/2	$-1/2$	0
x_1	1/2	1	1/2	0	1/6	1/6	0
x_3^r	1/2	0	3/2	0	$-3/2$	$-1/2$	1

We got the column for x_3 using the formulas

$$\widetilde{c}_3 = c_3 - (3/2, 1/2, -1)\begin{pmatrix} -3 \\ 6 \\ 0 \end{pmatrix} = 0 + 9/2 - 6/2 = 3/2,$$

and $\widetilde{A}_3 = A_B^{-1}A_3 = \begin{pmatrix} 1/2 & -1/2 & 0 \\ 1/6 & 1/6 & 0 \\ -3/2 & -1/2 & 1 \end{pmatrix}\begin{pmatrix} -3 \\ 6 \\ 0 \end{pmatrix} = \begin{pmatrix} -9/2 \\ 1/2 \\ 3/2 \end{pmatrix}$.

Pivoting on x_3-column and Row 3 we get

		x_1	x_3	x_4	x_1^r	x_2^r	x_3^r
$y_0 = -\xi$	0	0	0	0	1	1	1
x_4	2	0	0	1	-4	-2	3
x_1	1/3	1	0	0	2/3	1/3	$-1/3$
x_3	1/3	0	1	0	-1	$-1/3$	2/3

So, vector $(\frac{19}{3}, \frac{8}{3}, -\frac{14}{3})^T$ indeed is an optimal vector in **D** and the corresponding optimal vector in **P** is $(1/3, 0, 1/3, 2)^T$. The optimal cost is $-10/3$.