# An example of the revised 2-phase simplex method - CORRECTED

Suppose we are given the problem

$$\text{Maximize } z = 19x_1 + 13x_2 + 12x_3 + 17x_4$$

subject to

$$\begin{cases} 3x_1 & +2x_2 & +x_3 & +2x_4 & = & 225, \\ x_1 & +x_2 & +x_3 & +x_4 & = & 117, \\ 4x_1 & +3x_2 & +3x_3 & +4x_4 & = & 420 \\ x_1, & x_2, & x_3, & x_4 & \geq & 0. \end{cases} \tag{1}$$

Add to each of the equations its own variable $y_i$ and consider the auxiliary problem of the maximization of $\xi = -y_1 - y_2 - y_3$. Adding each equation from Row 0 we get the following tableau.

| | | $y_1$ | $y_2$ | $y_3$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|---|---|---|---|---|
| $y_0 = -\xi$ | 762 | 0 | 0 | 0 | 8 | 6 | 5 | 7 |
| $y_1$ | 225 | 1 | 0 | 0 | 3 | 2 | 1 | 2 |
| $y_2$ | 117 | 0 | 1 | 0 | 1 | 1 | 1 | 1 |
| $y_3$ | 420 | 0 | 0 | 1 | 4 | 3 | 3 | 4 |

The first four columns of this tableau form our matrix CARRY-0. Following Bland's Rule, the pivot column corresponds to $x_1$. The best ratio is in Row 1. Pivoting, we calculate only elements in the first four columns. Our CARRY-1 is

| | | $y_1$ | $y_2$ | $y_3$ |
|---|---|---|---|---|
| $y_0 = -\xi$ | 162 | $-8/3$ | 0 | 0 |
| $x_1$ | 75 | $1/3$ | 0 | 0 |
| $y_2$ | 42 | $-1/3$ | 1 | 0 |
| $y_3$ | 120 | $-4/3$ | 0 | 1 |

Now we calculate $\overline{d}_j$ using the formula

$$\overline{d}_j = d_j - \pi^T A_j, \tag{2}$$

where $-\pi^T$ is the vector in the last 3 entries of Row 0, and $A_j$ is the $j$th column of the original matrix $A$. Since $x_1$ is in the basis, we first try $\overline{d}_2$:

$$\overline{d}_2 = 6 + (-8/3,\ 0,\ 0) \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = 6 - 16/3 = 2/3 > 0.$$

So we will pivot on $x_2$. We calculate the column $\widetilde{A}_2$ using the formula

$$\widetilde{A}_j = A_B^{-1} A_j, \tag{3}$$

where $A_B^{-1}$ is formed by the last 3 columns and 3 rows of the last tableau. We have $\widetilde{A}_2 = \begin{pmatrix} 1/3 & 0 & 0 \\ -1/3 & 1 & 0 \\ -4/3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/3 \\ 1/3 \end{pmatrix}$. Adding column $\begin{pmatrix} 2/3 \\ 2/3 \\ 1/3 \\ 1/3 \end{pmatrix}$ to the last tableau and pivoting on the first row we get CARRY-2:

| | | $y_1$ | $y_2$ | $y_3$ |
|---|---|---|---|---|
| $y_0 = -\xi$ | 87 | $-3$ | 0 | 0 |
| $x_2$ | 225/2 | 1/2 | 0 | 0 |
| $y_2$ | 9/2 | $-1/2$ | 1 | 0 |
| $y_3$ | 165/2 | $-3/2$ | 0 | 1 |

Since $x_2$ is in the basis and $x_1$ just got out of it, we first calculate $\overline{d}_3$:

$$\overline{d}_3 = 5 + (-3,\, 0,\, 0) \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = 2 > 0.$$

Then similarly to above $\widetilde{A}_3 = \begin{pmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ -3/2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 3/2 \end{pmatrix}$. Adding column $\begin{pmatrix} 2 \\ 1/2 \\ 1/2 \\ 3/2 \end{pmatrix}$ to the last tableau and pivoting on the second row we get CARRY-3:

| | | $y_1$ | $y_2$ | $y_3$ |
|---|---|---|---|---|
| $y_0 = -\xi$ | 69 | $-1$ | $-4$ | 0 |
| $x_2$ | 108 | 1 | $-1$ | 0 |
| $x_3$ | 9 | $-1$ | 2 | 0 |
| $y_3$ | 69 | 0 | $-3$ | 1 |

Now we should check $\overline{d}_1$ again and it turns out to be positive:

$$\overline{d}_1 = 8 + (-1,\, -4,\, 0) \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = 1 > 0.$$

Then $\widetilde{A}_1 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & -3 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$. Adding column $\begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix}$ to the last tableau and pivoting on the first row we get CARRY-4:

| | | $y_1$ | $y_2$ | $y_3$ |
|---|---|---|---|---|
| $y_0 = -\xi$ | 15 | $-3/2$ | $-7/2$ | 0 |
| $x_1$ | 54 | $1/2$ | $-1/2$ | 0 |
| $x_3$ | 63 | $-1/2$ | $3/2$ | 0 |
| $y_3$ | 15 | $-1/2$ | $-5/2$ | 1 |

Note that $x_1$ first entered the basis, then exited it, and now entered again. Since $x_1$ and $x_3$ are in the basis and $x_2$ was just removed from the basis, we know $x_1$,$x_2$ and $x_3$ will not enter the basis on this step. So we calculate $\overline{d}_4$:

$$\overline{d}_4 = 7 + (-3/2,\ -7/2,\ 0)\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = 1/2 > 0.$$

Then $\widetilde{A}_4 = \begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 3/2 & 0 \\ -1/2 & -5/2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$. Adding column $\begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$ to the last tableau and pivoting on the last row we get CARRY-5:

| | | $y_1$ | $y_2$ | $y_3$ |
|---|---|---|---|---|
| $y_0 = -\xi$ | 0 | $-1$ | $-1$ | $-1$ |
| $x_1$ | 39 | 1 | 2 | $-1$ |
| $x_3$ | 48 | 0 | 4 | $-1$ |
| $x_4$ | 30 | $-1$ | $-5$ | 2 |

Thus we found a basic feasible solution of the original problem. Now we replace the row $-\mathbf{d}_B^T A_B^{-1}$ (the last 3 entries) by the row

$$-\mathbf{c}_B^T A_B^{-1} = -(19,\ 12,\ 17)\begin{pmatrix} 1 & 2 & -1 \\ 0 & 4 & -1 \\ -1 & -5 & 2 \end{pmatrix} = (-2,\ -1,\ -3).$$

Also, the current value of $-z$ is

$$-\mathbf{c}_B^T A_B^{-1}\mathbf{b} = (-2,\ -1,\ -3)\begin{pmatrix} 225 \\ 117 \\ 420 \end{pmatrix} = -1827.$$

Hence, our CARRY-6 is

| | | $y_1$ | $y_2$ | $y_3$ |
|---|---|---|---|---|
| $x_0 = -z$ | $-1827$ | $-2$ | $-1$ | $-3$ |
| $x_1$ | 39 | 1 | 2 | $-1$ |
| $x_3$ | 48 | 0 | 4 | $-1$ |
| $x_4$ | 30 | $-1$ | $-5$ | 2 |

Since $x_1$, $x_3$ and $x_4$ are in the basis we only need to check column 2. $\bar{c}_2 = 13 + (-2, -1, -3)\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = -14 < 0$, Since $\bar{c}_2$ is negative, the optimal value is 1827 attained at (39, 0, 48, 30).