An example of the dual simplex method

Suppose we are given the problem

$$\text{Maximize } z = -2x_1 - 3x_2 - 4x_3 - 5x_4$$

$$\text{subject to} \quad \begin{cases} x_1 & -x_2 & +x_3 & -x_4 & \geq & 10, \\ x_1 & -2x_2 & +3x_3 & -4x_4 & \geq & 6, \\ 3x_1 & -4x_2 & +5x_3 & -6x_4 & \geq & 15 \\ x_1, & x_2, & x_3, & x_4 & \geq & 0. \end{cases} \tag{1}$$

If we would have inequalities $\leq$ instead of $\geq$, then the usual simplex would work nicely. The two-phase method is more tedious. But since all coefficients in $z = -2x_1 - 3x_2 - 4x_3 - 5x_4$ are non-positive, we are fine for the dual simplex.

Multiply the equations by -1 and add to each of the equations its own slack variable. Then we get the following tableau.

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
$x_0 = -z$	0	-2	-3	-4	-5	0	0	0
x_5	-10	-1	1	-1	1	1	0	0
x_6	-6	-1	2	-3	4	0	1	0
x_7	-15	-3	4	-5	6	0	0	1

Choose Row 1 to pivot on. The ratio for x_1 ($\frac{-2}{-1} = 2$) is less than the ratio for x_3 ($\frac{-4}{-1} = 4$), so pivot on $a_{1,1}$. After pivoting, we get

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
$x_0 = -z$	20	0	-5	-2	-7	-2	0	0
x_1	10	1	-1	1	-1	-1	0	0
x_6	4	0	1	-2	3	-1	1	0
x_7	15	0	1	-2	3	-3	0	1

Now every $a_{i,0}$ for $i > 0$ is nonnegative. So, the tableau is optimal. But suppose that the boss adds now the new restriction:

$$x_1 + 2x_2 + 3x_3 - 4x_4 \leq 8.$$

With the dual simplex, we do not need to start from scratch. We simply add the new row and one more column to our final tableau.

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
$x_0 = -z$	20	0	-5	-2	-7	-2	0	0	0
x_1	10	1	-1	1	-1	-1	0	0	0
x_6	4	0	1	-2	3	-1	1	0	0
x_7	15	0	1	-2	3	-3	0	1	0
x_8	8	1	2	3	-4	0	0	0	1

Excluding from the last row x_1, x_6 and x_7, we get the tableau

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
$x_0 = -z$	20	0	-5	-2	-7	-2	0	0	0
x_1	10	1	-1	1	-1	-1	0	0	0
x_6	4	0	1	-2	3	-1	1	0	0
x_7	15	0	1	-2	3	-3	0	1	0
x_8	-2	0	3	2	-3	1	0	0	1

Note that if in the last row there were no -3, then the LP would be infeasible. Now we pivot on x_4:

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
$x_0 = -z$	$74/3$	0	-12	$-20/3$	0	$-13/3$	0	0	$-7/3$
x_1	$32/3$	1	-2	$1/3$	0	$-4/3$	0	0	$-1/3$
x_6	2	0	4	0	0	0	1	0	1
x_7	13	0	4	0	0	-2	0	1	1
x_4	$2/3$	0	-1	$-2/3$	1	$-1/3$	0	0	$-1/3$

Thus we got a solution again.