# Second example of the simplex method

Suppose we are given the problem

$$\text{Maximize } z = x_1 - x_2 + x_3$$

subject to

$$(1)\qquad \begin{cases} 2x_1 - x_2 + 2x_3 + x_4 = 4 \\ 2x_1 - 3x_2 + x_3 + x_5 = -5 \\ -x_1 + x_2 - 2x_3 + x_6 = -1 \\ x_1, \ x_2, \ x_3, \ x_4, \ x_5 \ x_6 \geq 0. \end{cases}$$

This system is solved with respect to $x_4, x_5$, and $x_6$, but the obtained basic solution is not feasible. So, we will look for a feasible solution by solving another linear program obtained as follows.

Multiply the last two equations by $-1$ in order to get positive RHS, then add to either of these equations its own variable and switch the LHS with the RHS:

$$(2)\qquad \begin{cases} 4 = 2x_1 - x_2 + 2x_3 + x_4 \\ 5 = -2x_1 + 3x_2 - x_3 - x_5 + y_1 \\ 1 = x_1 - x_2 + 2x_3 - x_6 + y_2 \\ x_1, \ x_2, \ x_3, \ x_4, \ x_5, \ x_6, \ y_1, \ y_2 \geq 0. \end{cases}$$

Note that a basic feasible solution of system (2) with $y_1 = y_2 = 0$ would be a basic feasible solution of (1). So, in search of such solutions, we will attempt to minimize $\xi = y_1 + y_2$ under conditions (2). A good feature is that we already have the following basic feasible solution of (2): $x_1 = x_2 = x_3 = x_5 = x_6 = 0$, $x_4 = 4$, $y_1 = 5$, $y_2 = 1$. Consider the tableau corresponding to our new linear program:

| | | $y_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $y_1$ | $y_2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $y_0 = -\xi$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | $-1$ | $-1$ |
| $x_4$ | 4 | 0 | 2 | $-1$ | 2 | 1 | 0 | 0 | 0 | 0 |
| $y_1$ | 5 | 0 | $-2$ | 3 | $-1$ | 0 | $-1$ | 0 | 1 | 0 |
| $y_2$ | 1 | 0 | 1 | $-1$ | 2 | 0 | 0 | $-1$ | 0 | 1 |

We cannot yet start pivoting, since the coefficients at basic variables $y_1$ and $y_2$ in Row 0 are non-zeros. Excluding $y_1$ and $y_2$ from Row 0, we get

| | | $y_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $y_1$ | $y_2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $y_0 = -\xi$ | 6 | 1 | -1 | 2 | 1 | 0 | -1 | -1 | 0 | 0 |
| $x_4$ | 4 | 0 | 2 | $-1$ | 2 | 1 | 0 | 0 | 0 | 0 |
| $y_1$ | 5 | 0 | $-2$ | 3 | $-1$ | 0 | $-1$ | 0 | 1 | 0 |
| $y_2$ | 1 | 0 | 1 | $-1$ | 2 | 0 | 0 | $-1$ | 0 | 1 |

Choose Column $x_3$ as pivot column. Then the pivot row will be Row 3:

| | | $y_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $y_1$ | $y_2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $y_0 = -\xi$ | $11/2$ | $1$ | $3/2$ | $5/2$ | $0$ | $0$ | $1$ | $1/2$ | $0$ | $1/2$ |
| $x_4$ | $3$ | $0$ | $1$ | $0$ | $0$ | $1$ | $0$ | $1$ | $0$ | $-1$ |
| $y_1$ | $11/2$ | $0$ | $-3/2$ | $5/2$ | $0$ | $0$ | $-1$ | $-1/2$ | $1$ | $1/2$ |
| $x_3$ | $1/2$ | $0$ | $1/2$ | $-1/2$ | $1$ | $0$ | $0$ | $-1/2$ | $0$ | $1/2$ |

Now we pivot on $x_2$:

| | | $y_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $y_1$ | $y_2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $y_0 = -\xi$ | $0$ | $1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $-1$ | $-1$ |
| $x_4$ | $3$ | $0$ | $1$ | $0$ | $0$ | $1$ | $0$ | $1$ | $1$ | $-1$ |
| $x_2$ | $11/5$ | $0$ | $-3/5$ | $1$ | $0$ | $0$ | $-2/5$ | $-1/5$ | $2/5$ | $1/5$ |
| $x_3$ | $16/10$ | $0$ | $2/10$ | $0$ | $1$ | $0$ | $-1/5$ | $-6/10$ | $1/5$ | $6/10$ |

Thus we found a basic feasible solution of (1) and return to this problem: Delete the columns corresponding to $y_1$ and $y_2$, and replace the objective function.

| | | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|---|---|
| $x_0 = -z$ | $0$ | $1$ | $1$ | $-1$ | $1$ | $0$ | $0$ | $0$ |
| $x_4$ | $3$ | $0$ | $1$ | $0$ | $0$ | $1$ | $0$ | $1$ |
| $x_2$ | $11/5$ | $0$ | $-3/5$ | $1$ | $0$ | $0$ | $-2/5$ | $-1/5$ |
| $x_3$ | $16/10$ | $0$ | $2/10$ | $0$ | $1$ | $0$ | $-1/5$ | $-6/10$ |

Excluding basic variables $x_2$ and $x_3$ from Row 0, we get

| | | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|---|---|
| $x_0 = -z$ | $3/5$ | $1$ | $1/5$ | $0$ | $0$ | $0$ | $-1/5$ | $2/5$ |
| $x_4$ | $3$ | $0$ | $1$ | $0$ | $0$ | $1$ | $0$ | $1$ |
| $x_2$ | $11/5$ | $0$ | $-3/5$ | $1$ | $0$ | $0$ | $-2/5$ | $-1/5$ |
| $x_3$ | $16/10$ | $0$ | $2/10$ | $0$ | $1$ | $0$ | $-1/5$ | $-6/10$ |

Choose $x_6$ as the pivot column. Then the pivot row is Row 1. After the pivot we have

| | | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|---|---|
| $x_0 = -z$ | $-3/5$ | $1$ | $-1/5$ | $0$ | $0$ | $-2/5$ | $-1/5$ | $0$ |
| $x_6$ | $3$ | $0$ | $1$ | $0$ | $0$ | $1$ | $0$ | $1$ |
| $x_2$ | $14/5$ | $0$ | $-2/5$ | $1$ | $0$ | $1/5$ | $-2/5$ | $0$ |
| $x_3$ | $17/5$ | $0$ | $4/5$ | $0$ | $1$ | $3/5$ | $-1/5$ | $0$ |

This tableau corresponds to the basic solution $x_5 = x_1 = x_4 = 0$, $x_2 = 14/5$, $x_3 = 17/5$, $x_6 = 3$, which gives $-z = 3/5$. Since we do not have negative entries in Row 0, this solution is optimal.