Two phase simplex

- Suppose we are given the problem

$$\text{Minimize } z = -x_1 + x_2 - x_3$$

subject to

$$\begin{array}{cccccccc} 2x_1 & -x_2 & +2x_3 & +x_4 & & & = & 4 \\ 2x_1 & -3x_2 & +x_3 & & +x_5 & & = & -5 \\ -x_1 & +x_2 & -2x_3 & & & +x_6 & = & -1 \\ x_1, & x_2, & x_3, & x_4, & x_5 & x_6 & \geq & 0. \end{array} \quad (1)$$

- This system is solved with respect to x_4, x_5, and x_6, but the obtained basic solution is not feasible. So, we will look for a feasible solution by solving another linear program obtained as follows.

- Multiply the last two equations by -1 in order to get positive RHS, then add to either of these equations its own variable and switch the LHS with the RHS:

$$\begin{array}{cccccccccc} 4 & = & 2x_1 & -x_2 & +2x_3 & +x_4 & & & & \\ 5 & = & -2x_1 & +3x_2 & -x_3 & & -x_5 & & +y_1 & \\ 1 & = & x_1 & -x_2 & +2x_3 & & & -x_6 & & +y_2 \\ & & x_1, & x_2, & x_3, & x_4, & x_5, & x_6, & y_1, & y_2 \geq 0. \end{array} \quad (2)$$

- Note that a basic feasible solution of system (2) with $y_1 = y_2 = 0$ would be a basic feasible solution of (1).
- So, in search of such solutions, we will attempt to minimize $\xi = y_1 + y_2$ under conditions (2).
- We have added y_1 and y_2 so that we have the following basic feasible solution of (2): $x_1 = x_2 = x_3 = x_5 = x_6 = 0$, $x_4 = 4$, $y_1 = 5$, $y_2 = 1$.
- We could also add a variable for the first row, but we don't have to since it is already solved for x_4.

- Consider the tableau corresponding to our new linear program:

		x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2
$-\xi$	0	0	0	0	0	0	0	1	1
x_4	4	2	-1	2	1	0	0	0	0
y_1	5	-2	3	-1	0	-1	0	1	0
y_2	1	1	-1	2	0	0	-1	0	1

- We cannot yet start pivoting, since the coefficients at basic variables y_1 and y_2 in Row 0 are non-zeros. Excluding y_1 and y_2 from Row 0, we get...

		x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2
$-\xi$	-6	1	-2	-1	0	1	1	0	0
x_4	4	2	-1	2	1	0	0	0	0
y_1	5	-2	3	-1	0	-1	0	1	0
y_2	1	1	-1	2	0	0	-1	0	1

Choose Column x_3 as the pivot column. Then the pivot row will be Row 3:

		x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2
$-\xi$	$-11/2$	$3/2$	$-5/2$	0	0	1	$1/2$	0	$1/2$
x_4	3	1	0	0	1	0	1	0	-1
y_1	$11/2$	$-3/2$	$5/2$	0	0	-1	$-1/2$	1	$1/2$
x_3	$1/2$	$1/2$	$-1/2$	1	0	0	$-1/2$	0	$1/2$

Now we pivot on column x_2 and Row 2:

		x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2
$-\xi$	0	0	0	0	0	0	0	1	1
x_4	3	1	0	0	1	0	1	1	-1
x_2	$11/5$	$-3/5$	1	0	0	$-2/5$	$-1/5$	$2/5$	$1/5$
x_3	$16/10$	$2/10$	0	1	0	$-1/5$	$-6/10$	$1/5$	$6/10$

- This is an optimal tableau for the auxiliary problem
- If the value of the objective function at the optimum was greater than 0, then ...
- we can could conclude that the original problem was infeasible.
- But it is 0, so we have found a bfs of the original problem
- Delete the columns corresponding to y_1 and y_2, and replace the original objective function.
- One special case that could happen:
    - If the optimum to the auxiliary problem is degenerate, an artifical variable, y_1 or y_2, could still be in the basis.
    - To *drive the artifical variables out of the the basis* if row i is solved for, say y_1 (so y_1 is still in the basis), then pivot on entry (i, j) for any of the original columns j such that $a_{i,j} \neq 0$ (even if $\overline{c}_j > 0$ or $a_{i,j} < 0$).
    - Repeat until no artifical variables remain in the basis. (See page 56 of the book).

- Here is the new tableau - the top row corresponds exactly to the original objective function

		x_1	x_2	x_3	x_4	x_5	x_6
$-z$	0	-1	1	-1	0	0	0
x_4	3	1	0	0	1	0	1
x_2	11/5	$-3/5$	1	0	0	$-2/5$	$-1/5$
x_3	16/10	2/10	0	1	0	$-1/5$	$-6/10$

- Excluding basic variables x_2 and x_3 from Row 0, we get

		x_1	x_2	x_3	x_4	x_5	x_6
$-z$	$-3/5$	$-1/5$	0	0	0	1/5	$-2/5$
x_4	3	1	0	0	1	0	1
x_2	11/5	$-3/5$	1	0	0	$-2/5$	$-1/5$
x_3	16/10	2/10	0	1	0	$-1/5$	$-6/10$

- Choose x_6 as the pivot column. Then the pivot row is Row 1. After the pivot we have

		x_1	x_2	x_3	x_4	x_5	x_6
$-z$	3/5	1/5	0	0	2/5	1/5	0
x_6	3	1	0	0	1	0	1
x_2	14/5	$-2/5$	1	0	1/5	$-2/5$	0
x_3	17/5	4/5	0	1	3/5	$-1/5$	0

This tableau corresponds to the basic solution $x_5 = x_1 = x_4 = 0$, $x_2 = 14/5$, $x_3 = 17/5$, $x_6 = 3$, which gives $-z = 3/5$. Since we do not have negative entries in Row 0, this solution is optimal.