Basic easy case of the simplex method

Suppose we are given the problem

$$\text{Minimize } z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

subject to
$$\begin{cases} a_{1,1}x_1 & +a_{1,2}x_2 & +\ldots & +a_{1,n}x_n & = & b_1, \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m,1}x_1 & +a_{m,2}x_2 & +\ldots & +a_{m,n}x_n & = & b_m, \\ x_1, & x_2, & \ldots, & x_n & \geq & 0. \end{cases}$$

Let $A = \{a_{i,j}\}_{1\leq i\leq m, 1\leq j\leq n}$, $b = (b_1, \ldots, b_m)^T$ and $b = (c_1, \ldots, c_n)^T$. We may assume that the rank of A is m, since otherwise we simply delete some dependent equations. Rewrite the objective function as a new equation with the new variable $x_0 = -z$, solve the new system with respect to some variables (including x_0) (for simplicity, we assume that we were able to solve it with respect to $x_0, \ldots, x_m$) and switch the LHS with the RHS.

$$\begin{cases} b'_0 & = & x_0 & +0 & +0 & \ldots & +0 & +a'_{0,m+1}x_{m+1} & \ldots & +a'_{0,n}x_n \\ b'_1 & = & 0 & +x_1 & +0 & \ldots & +0 & +a'_{1,m+1}x_{m+1} & \ldots & +a'_{1,n}x_n \\ b'_2 & = & 0 & +0 & +x_2 & \ldots & +0 & +a'_{2,m+1}x_{m+1} & \ldots & +a'_{2,n}x_n \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ b'_m & = & 0 & +0 & \ldots & +0 & +x_m & a'_{m,m+1}x_{m+1} & \ldots & +a'_{m,n}x_n \\ & x_1, & x_2, & \ldots & x_n & \geq & 0. \end{cases}$$

Note that we excluded variables $x_1, \ldots, x_m$ from the 0th equation. Here we consider the very good case when

$$b'_i \geq 0 \quad \textbf{for every} \quad i = 1, \ldots, m.$$

In this case we have the basic feasible solution

$$x_j = \begin{cases} b'_j, & \text{if } 1 \leq j \leq m; \\ 0, & \text{otherwise.} \end{cases}$$

It gives the value $z = -x_0 = -b'_0$. Write the tableau of the coefficients for this system replacing b'_i with $a_{i,0}$ and $a'_{i,j}$ with $a_{i,j}$ (for simplicity):

		x_0	x_1	x_2	$\ldots$	x_m	x_{m+1}	$\ldots$	x_n
$x_0 = -z$	$a_{0,0}$	1	0	0	$\ldots$	0	$a_{0,m+1}$	$\ldots$	$a_{0,n}$
x_1	$a_{1,0}$	0	1	0	$\ldots$	0	$a_{1,m+1}$	$\ldots$	$a_{1,n}$
x_2	$a_{2,0}$	0	0	1	$\ldots$	0	$a_{2,m+1}$	$\ldots$	$a_{2,n}$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
x_m	$a_{m,0}$	0	0	0	$\ldots$	1	$a_{m,m+1}$	$\ldots$	$a_{m,n}$

In our case Row i is lexicographically positive for each $i \geq 1$. We describe a general step not assuming that the system is solved with respect to $x_1, \ldots, x_m$. We only will assume that it is solved with respect to some variables and Row i is lexicographically positive for each $i \geq 1$. And we will assume that Equation i is solved with respect to x_{j_i}.

First observe that we never will get rows consisting only of zeros, since we assumed that the rank of our system is m, and the rank will not change because we are only performing elementary row operations.

STEP 1. Let $a_{0,s} = \min\{a_{0,j} \;:\; 1 \le j \le n\}$. If $a_{0,s} \ge 0$, then $z = -a_{0,0} + \sum_{j=1}^{n} a_{0,j}x_j$ for any $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ such that $Ax = b$. Since we require $x \ge 0$, $z = -a_{0,0}$ is the optimal solution. Otherwise, pick any s such that $a_{0,s} < 0$ and let s be the pivot column.

STEP 2. If $a_{i,s} \le 0$ for all $1 \le i \le m$, then $x_0 = -z$ is not bounded from above. Indeed, for each $t > 0$ we can let

$$x_j = \begin{cases} t, & \text{if } j = s; \\ a_{i,0} - a_{i,s}t & \text{if } j = j_i,\ 1 \le i \le m; \\ 0, & \text{otherwise.} \end{cases}$$

Recall that in our case $a_{i,0} - a_{i,s}t \ge a_{i,0} \ge 0$. Every such assignment is a feasible solution to our system (please, check it) and

$$z = -a_{0,0} + ta_{0,s} \xrightarrow{t\to\infty} -\infty.$$

If there exists $a_{i,s} > 0$ then among rows R_i with $a_{i,s} > 0$ choose a row R_r such that the vector $\frac{1}{a_{r,s}}R_r$ is lexicographically minimum. Then $a_{r,s}$ is the pivot entry and R_r is the pivot row.

STEP 3. Use Gaussian elimination to exclude x_s from all rows apart from R_r and make the coefficient at x_s in R_r equal 1. Then go to Step 1.

Note that after Gaussian elimination we get a system equivalent to the original.

Lemma 1. *After Step 3,*
(a) All rows remain lexicographically positive;
(b) Row 0 *lexicographically increases;*

Proof. Each new row R_i' is equal to $R_i - \frac{a_{i,s}}{a_{r,s}}R_r$. Thus, if $a_{i,s} \le 0$, then we add to the lexicographically positive R_i another lexicographically nonnegative row, which results in a lexicographically positive row. Suppose that $a_{i,s} > 0$. Then

$$R_i' = R_i - \frac{a_{i,s}}{a_{r,s}}R_r = a_{i,s}\left[\frac{1}{a_{i,s}}R_i - \frac{1}{a_{r,s}}R_r\right].$$

The expression in [] is lexicographically nonnegative by the choice of r and cannot be zero-row, since the rank of our system is r. This proves (a).

Row 0 lexicographically increases, since we add to it lexicographically positive row R_r with the positive coefficient $-\frac{a_{0,s}}{a_{r,s}}$. □

Now by (a) we again can do Step 1 with the new system, and by (b) and the fact that Row 0 is determined by the basis, no basis will appear twice. Since the number of bases is at most $\binom{n}{m}$, after a finite number of steps, we will solve the program.