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Math 482 - Fall 2015 - Theodore Molla

1. Basic feasible solutions and the simplex method

1.1. **Basic feasible solutions.** Recall that a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ has rank r, if $\mathbf{A}$ if exactly r linearly independent rows and r linearly independent columns.

Assumption 1 (Initial assumptions)**.** Unless otherwise mentioned whenever we say:

$$\min \mathbf{c}^T\mathbf{x} \text{ subject to } \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0},$$

We mean that $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{c} \in \mathbb{R}^n$.

We also will assume that $\mathbf{A}\mathbf{x} = \mathbf{b}$ has at least one solution and that and that $\mathbf{A} \in \mathbb{R}^{m\times n}$ has rank m. Note that this implies that $n \geq m$ (In equational form, the matrix $\mathbf{A}$ is always at least as wide as it is tall.)

Why can we make these assumption? Also recall that it is "easy" (via Gaussian Elimination) to check if $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a solution. This handles the first assumption. The following proposition justifies the second assumption.

Proposition 1. *Suppose $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$ and that there exists a solution to the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$. If $\mathbf{A}$ does not have rank m, then there exists $i_0 \in [m]$, such that when*

- *$\mathbf{A}'$ is obtained from $\mathbf{A}$ by removing row i_0,*
- *$\mathbf{b}'$ is obtained from $\mathbf{b}$ by removing the i_0th entry of $\mathbf{b}$,*

$F := \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$ is equal to $F' := \{\mathbf{x} : \mathbf{A}'\mathbf{x} = \mathbf{b}', \mathbf{x} \geq 0\}$.

Proof. We clearly have that every $\mathbf{x} \in F$ is also F'. This is because $\mathbf{A}\mathbf{x} = \mathbf{b}$ implies that for every $i \in [m] \setminus \{i_0\}$, $\mathbf{a}_i^T\mathbf{x} = \mathbf{b}_i$, which is equivalent to the expression $\mathbf{A}'\mathbf{x} = \mathbf{b}'$.

Since $\mathbf{A}$ does not have rank m, there exists $\mathbf{y} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$ such that $\mathbf{y}^T\mathbf{A} = \mathbf{0}^T$. Since $\mathbf{y} \neq \mathbf{0}$, we can assume without loss of generality that that $y_m \neq 0$. So we have that

$$y_m\mathbf{a}_m^T = -\sum_{i=1}^{m-1} y_i\mathbf{a}_i^T.$$

and, since there exists $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{A}\mathbf{x} = \mathbf{b}$, we have that

$$y_m b_m = y_m\mathbf{a}_m^T\mathbf{x} = -\sum_{i=1}^{m-1} y_i\mathbf{a}_i^T\mathbf{x} = -\sum_{i=1}^{m-1} y_i b_i.$$

Now let $\mathbf{x}' \in F'$. Recall that we will complete the proof if we show that $\mathbf{x}' \in F$. Therefore, the only thing we need to show is that

$$\mathbf{a}_m^T\mathbf{x} = b_m,$$

and this follows from the following expression and the fact that $y_m \neq 0$:

$$y_m\mathbf{a}_m^T\mathbf{x} = -\sum_{i=1}^{m-1} y_i\mathbf{a}_i^T\mathbf{x} = -\sum_{i=1}^{m-1} y_i b_i = y_m b_m.$$

□

Definition 1 (Basic Feasible Solution (bfs))**.** Let

$$\min \mathbf{c}^T\mathbf{x} \text{ subject to } \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0},$$

and let

$$F := \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}; \mathbf{x} \geq \mathbf{0}\}.$$

Recall that we assume that $\mathbf{A}$ has rank m. An index set $B \subseteq [n]$ is a *basis* if $|B| = m$ and $\mathbf{A}_B$ is non-singular (a non-singular matrix, is an invertible matrix, or a matrix with non-zero determinant, etc.) A basis B is a *feasible basis* if $|B| = m$ and $\mathbf{A}_B^{-1}b \geq \mathbf{0}$. A feasible solution $\mathbf{x}$ is a *basic feasible solution (bfs)*, if there exists a basis B such that $x_j = 0$ for every $j \in \overline{B}$ (Here $\overline{B} := \{j \in [j] : j \notin B\}$). However, it is NOT necessarily true that if $\mathbf{x}$ is a bfs, then $x_j > 0$ for every $j \in B$. We use these term in reference to an LP and also to the set of feasible solutions F, i.e. we may say that $\mathbf{x}$ is a basic feasible solution of F or a basic feasible solution of the LP.

Proposition 2. *A feasible basis B corresponds to exactly one basic feasible solution. This basic feasible solution $\mathbf{x}$ is defined by $\mathbf{x}_B = \mathbf{A}_B^{-1}b$ and $\mathbf{x}_{\overline{B}} = \mathbf{0}$. In particular, if $\mathbf{x}'$ is feasible and $\mathbf{x}' \neq \mathbf{x}$, then there exists $j \in \overline{B}$ such that $x'_j > 0$.*

Proof. Suppose $\mathbf{x}$ is a basic feasible solution corresponding to B. By definition, it must be that $\mathbf{x}_{\overline{B}} = \mathbf{0}$, and this implies that $b = \mathbf{A}\mathbf{x} = \mathbf{A}_B\mathbf{x}_B$, so $\mathbf{x}_B = \mathbf{A}_B^{-1}b$. The fact that $\mathbf{x}_{\overline{B}} = \mathbf{0}$ uniquely determined $\mathbf{x}$, so if $\mathbf{x}' \neq \mathbf{x}$, we must have $\mathbf{x}'_{\overline{B}} \neq \mathbf{0}$. ☐

Lemma 3 ([2] Theorem 2.2)**.** *If $\mathbf{x}$ is a bfs of*

$$F := \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}; \mathbf{x} \geq \mathbf{0}\}$$

with associated basis B, then there exists $\mathbf{c}$ such that $\mathbf{x}$ is the unique optimal solution of

$$\min \mathbf{c}^T\mathbf{x} \textit{ subject to } \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}.$$

Proof. Define $\mathbf{c}$ by $\mathbf{c}_B = 0$ and $c_j = 1$ for all $j \in \overline{B}$. By the definition of a bfs, $\mathbf{c}^T\mathbf{x} = 0$. Now let $\mathbf{x}'$ be a feasible vector such that $\mathbf{x}' \neq \mathbf{x}$. By Proposition 2 there exist $j \in \overline{B}$ such that $x'_j > 0$, so since $\mathbf{x}' \geq 0$, $\mathbf{c}^T\mathbf{x}' \geq c_j x'_j = x'_j > \mathbf{c}^T\mathbf{x}$. ☐

Proposition 4 ([1] Theorem 2.1)**.** *A feasible solution $\mathbf{x}$ to the linear program*

$$\min \mathbf{c}^T\mathbf{x} \textit{ subject to } \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0},$$

is a bfs if and only if the columns of the matrix $\mathbf{A}_K$ are linearly independent where

$$K := \{j \in [n] : x_j > 0\}.$$

Proof. If $\mathbf{x}$ is a bfs, then there exists a basis $B \subseteq [n]$, such that $K \subseteq B$. The fact that $\mathbf{A}_B$ is linearly independent implies that $\mathbf{A}_K$ is linearly independent.

To prove the other direction, assume $\mathbf{A}_K$ is linearly independent. Pick $B \subseteq [n]$ as large as possible such that:

- $K \subseteq B \subseteq [n]$,
- the columns of $\mathbf{A}_B$ are linearly independent, and
- $x_j = 0$ for all $j \in [n] \setminus B$.

Clearly, $|B| \leq m$, since $\mathbf{A}_B$ is linearly independent. If $|B| < m$, then since $\mathbf{A}$ has rank m, it must be that there exists $j \in [n] \setminus B$ such that the columns of $\mathbf{A}_{B \cup \{j\}}$ are linearly independent, but this contradicts our selection of B ($B \cup \{j\}$ also satisfies the three conditions we placed on B, but has more element than B). Therefore, it must be that $|B| = m$, and this implies that $\mathbf{x}$ is a bfs. ☐

Lemma 5 (Theorem 2.1 in [2])**.** *Consider the linear program*

$$\min \mathbf{c}^T\mathbf{x} \textit{ subject to } \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}.$$

(1) If there is a feasible solution, then there is a basic feasible solution.
(2) If the linear program is bounded and there is a feasible solution $\mathbf{x}$, then there exists a basic feasible solution x_0 such that $\mathbf{c}^T\mathbf{x}_0 \leq \mathbf{c}^T\mathbf{x}$.

Proof. Let $\mathbf{x}$ be a feasible solution. Assume that $\mathbf{x}$ is not a basic feasible solution as other was there is nothing to prove for both parts (1) and (2). Let

$$F' := \{\mathbf{x}' : \ \mathbf{x}' \text{ is a feasible solution and } x'_j = 0 \text{ for all } j \in [n] \text{ such that } x_j = 0\}.$$

First we will prove (1). Let $\overline{\mathbf{x}}$ be an element of F' with the most zero entries [1]. Suppose $\overline{\mathbf{x}}$ is not a bfs let $K := \{j \in [n] : \overline{x}_j > 0\}$. By Proposition 4, the columns of $\mathbf{A}_K$ are not linearly independent, so there exists $\mathbf{w} \in R^n \setminus \{\mathbf{0}\}$ such that $\mathbf{w}_{\overline{K}} = \mathbf{0}$ and $\mathbf{A}_K\mathbf{w}_K = \mathbf{0}$. By construction we have that $\mathbf{A}\mathbf{w} = \mathbf{0}$, i.e. $\mathbf{w}$ is in he null space of $\mathbf{A}$, so for every $\theta > 0$, if we define $\mathbf{x}^* = \mathbf{x} + \theta\mathbf{w}$, then

$$\mathbf{A}\mathbf{x}^* := \mathbf{A}\mathbf{x} + \theta\mathbf{A}\mathbf{x} = \mathbf{b}.$$

[1] There can be more than one such $\overline{\mathbf{x}}$

We can assume that there exist $j \in K$ such that $w_j < 0$, because otherwise we could have selected $-\mathbf{w}$ instead of $\mathbf{w}$. Since $\mathbf{w}_{\overline{K}} = \mathbf{0}$, we can pick θ so that $\mathbf{x}^* \in F'$ and $\mathbf{x}^*_j = 0$ for some $j \in K$ a contradiction to our choice of $\overline{\mathbf{x}}$. This proves (1).

For (2) we essentially do the same proof, except instead of picking $\overline{\mathbf{x}}$ to be an element of F' with the most zero entries, we pick $\overline{\mathbf{x}}$ to be an element of F' such that $\mathbf{c}^T\overline{\mathbf{x}} \leq \mathbf{c}^T\mathbf{x}$ and, subject to this additional condition, has the most zero entries. We construct $\mathbf{w}$ in exactly the same way except we can assume that, since the LP is bounded, that $\mathbf{c}^T\mathbf{w} \leq 0$ and there exists $j \in [n]$ such that $w_j < 0$. If $\mathbf{c}^T\mathbf{w} = 0$, then we can make this assumption since $\mathbf{w} \neq \mathbf{0}$ and either $\mathbf{w}$ or $-\mathbf{w}$ will work. If $\mathbf{c}^T\mathbf{w} < 0$ and $\mathbf{w} \geq 0$, then the LP is unbounded, because for every $\theta > 0$, $\mathbf{x}^* = \overline{\mathbf{x}} + \theta\mathbf{w}$ is feasible and as θ goes to infinity $\mathbf{c}^T\mathbf{x}^*$ goes to $-\infty$. So define

$$\mathbf{A}\mathbf{x}^* := \mathbf{A}\mathbf{x} + \theta\mathbf{A}\mathbf{x} = \mathbf{b}$$

again and note that for any $\theta > 0$,

$$\mathbf{c}^T\mathbf{x}^* = \mathbf{c}^T\overline{\mathbf{x}} + \theta\mathbf{c}^T\mathbf{w} \leq \mathbf{c}^T\overline{\mathbf{x}}.$$

Therefore, we can select θ so that $\mathbf{x}^*$ violates our choice of $\overline{\mathbf{x}}$. □

Theorem 6 (Fundamental Theorem)**.**

(1) If the LP has a feasible solution, then the LP has a basic feasible solution.
(2) If the LP has no optimal solution, then the LP is infeasible or unbounded.
(3) If the LP has an optimal solution, then it has an optimal basic feasible solution

Proof. The first statement follows directly from the first statment in Lemma 5 and third statement follows directly from the second statement in Lemma 5. To prove the second statement we use the contrapositive, i.e. we assume that the LP is not infeasible and and not unbounded, and then deduce that the LP has an optimal solution.

By Lemma 5 and the fact that we are assume that the LP is not infeasible, there exists basic feasible solutions. Since there are at most a finite number of basic feasible solutions, we can select $\mathbf{x}^*$ so that for every basic feasilble solution $\mathbf{x}'$, $\mathbf{c}^T\mathbf{x}^* \leq \mathbf{c}^T\mathbf{x}'$.

Now let $\mathbf{x}$ be any feasible solution. By Lemma 5, there exists a bfs $\mathbf{x}'$ such that $\mathbf{x}' \leq \mathbf{x}$. By our selection of $\mathbf{x}^*$, we have that

$$\mathbf{c}^T\mathbf{x}^* \leq \mathbf{c}^T\mathbf{x}' \leq \mathbf{c}^T\mathbf{x}.$$

Since $\mathbf{x}$ was an arbitrary feasible solution, we have proved that $\mathbf{x}^*$ is an optimal basic feasible solution. □

1.2. **Lemmas related to the simplex method.** If at a step of the simplex method the basis is B, the vector $\overline{\mathbf{c}}^T := \mathbf{c}^T - \mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{A}$ is in columns 1 through n of the top row of the tableau. The entry $\overline{c}_j$ is referred to as the *relative cost of column* j. The following theorem proves the significance of this vector.

Proposition 7. *Let* $\mathbf{x}$ *is a bfs with associated basis* B *and define* $\overline{\mathbf{c}}^T := \mathbf{c}^T - \mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{A}$*, then* $\mathbf{c}^T\mathbf{x}' = \mathbf{c}^T\mathbf{x} + \overline{\mathbf{c}}^T\mathbf{x}'$ *for any feasible solution* $\mathbf{x}'$*. In particular, we have that* $\mathbf{x}$ *is an optimal solution if* $\overline{\mathbf{c}}^T \geq 0$.

Proof. Let $\mathbf{x}'$ be a feasible solution. In particular, we have that $\mathbf{A}\mathbf{x}' = \mathbf{b}$. Also, note that by Proposition 2, $\mathbf{c}^T\mathbf{x} = \mathbf{c}_B^T\mathbf{x}_B = \mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{b}$. This with the definition of $\overline{\mathbf{c}}$ and linearity gives the following:

$$\begin{aligned}\mathbf{c}^T\mathbf{x} + \overline{\mathbf{c}}^T\mathbf{x}' &= \mathbf{c}^T\mathbf{x} + \mathbf{c}^T\mathbf{x}' - \mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{A}\mathbf{x}' \\ &= \mathbf{c}^T\mathbf{x} + \mathbf{c}^T\mathbf{x}' - \mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{b} \\ &= \mathbf{c}^T\mathbf{x} + \mathbf{c}^T\mathbf{x}' - \mathbf{c}^T\mathbf{x} \\ &= \mathbf{c}^T\mathbf{x}'.\end{aligned}$$

When $\overline{\mathbf{c}}^T \geq 0$, $\mathbf{x}$ is an optimum, because $\mathbf{c}^T\mathbf{x}' = \mathbf{c}^T\mathbf{x} + \overline{\mathbf{c}}^T\mathbf{x}' \geq \mathbf{c}^T\mathbf{x}$, since $\mathbf{x}' \geq 0$. □

In the simplex procedure there are two row operations: either we multiple some row $i \in \{1, \ldots, n\}$ by α, or we add some multiple of a row $i \in \{1, \ldots, n\}$ to a row $j \in \{0, \ldots, n\}$. The first operation corresponds to pre-multiplication of $\mathbf{T}$ by $\mathbf{I}(i, \alpha)$ which is obtained from the identity matrix $\mathbf{I}$ by replacing 1 in the entry (i, i) with α, and the second operation corresponds to pre-multiplication of $\mathbf{T}$ by $\mathbf{I}(i, j, \alpha)$ which is obtained from the identity matrix $\mathbf{I}$ by replacing 0 in the entry (i, j) with α. This means that $\widetilde{\mathbf{T}}$ is obtained from $\mathbf{T}$ by a sequence of pre-multiplications by such matrices. Note that in all cases the matrix we pre-multiply by has the first column $[1, 0, 0, \ldots, 0]^T$, i.e. we never add a multiple of row 0 to row i where $i \in \{1, \ldots, n\}$ and we never multiply row 0 by a constant. Therefore, the product of these matrices has first column $[1, 0, 0, \ldots, 0]^T$ since the product of two matrices with the zero column $[1, 0, 0, \ldots, 0]^T$ also has the zero column $[1, 0, 0, \ldots, 0]^T$.

Assume the current ordered basis is $B = (j_1, \ldots, j_m)$, the original matrix is $\mathbf{A}$ and the original cost vector is $\mathbf{c}$. We know that, in the current tableau, row i is solved for variable x_{j_i}, so in column j_i, we should have a 0 in the top row and in rows 1 through m we should have the ith standard basis vector. In other words, the columns j_1 through j_m in order of the current tableau have the form

$$\left[\begin{array}{c} \mathbf{0}^T \\ \hline \\ \mathbf{I}_m \end{array}\right],$$

where $\mathbf{I}_m$ is the $m \times m$ identity matrix. Therefore, if $\mathbf{X}$ is the matrix we must pre-multiply the original tableau, $\mathbf{T}$, by in order to the get the current tableau,

$$\mathbf{X}\mathbf{T}_B = \mathbf{X}\left[\begin{array}{c} \mathbf{c}_B^T \\ \hline \mathbf{A}_B \end{array}\right] = \left[\begin{array}{c} \mathbf{0}^T \\ \hline \\ \mathbf{I}_m \end{array}\right].$$

Hence, the following theorem gives $\mathbf{X}$ and the current tableau in terms of $\mathbf{A}$, B and $\mathbf{c}$

Theorem 8. *Let*

$$\mathbf{T} = \left[\begin{array}{c|c} f & \mathbf{c}^T \\ \hline \\ \mathbf{b} & \mathbf{A} \end{array}\right]$$

be a tableau, and let $B = (j_1, \ldots, j_m)$ be an ordered basis. Let $\mathbf{X}$ be a non-singular matrix with first column $[1, 0, \ldots, 0]^T$. If

$$\mathbf{X}\left[\begin{array}{c} \mathbf{c}_B^T \\ \hline \mathbf{A}_B \end{array}\right] = \left[\begin{array}{c} \mathbf{0}^T \\ \hline \\ \mathbf{I}_m \end{array}\right],$$

then $\mathbf{X} = \left(\begin{array}{c|c} 1 & -\pi^T \\ \hline \\ \mathbf{0} & \mathbf{A}_B^{-1} \end{array}\right)$, *where $\pi^T = \mathbf{c}_B^T\mathbf{A}_B^{-1}$. In particular,*

$$\mathbf{X}\mathbf{T} = \left[\begin{array}{c|c} f - \pi^T\mathbf{b} & \overline{\mathbf{c}}^T \\ \hline \\ \mathbf{A}_B^{-1}\mathbf{b} & \mathbf{A}_B^{-1}\mathbf{A} \end{array}\right].$$

where $\overline{\mathbf{c}}^T = \mathbf{c}^T - \pi^T\mathbf{A}$.

Proof. Let $\mathbf{X} = \left[\begin{array}{c|c} 1 & \mathbf{u}^T \\ \hline \mathbf{0} & \mathbf{U} \end{array}\right]$ so $\mathbf{X}\left[\begin{array}{c} \mathbf{c}_B \\ \hline \mathbf{A}_B \end{array}\right] = \left[\begin{array}{c} \mathbf{0}^T \\ \hline \\ \mathbf{I} \end{array}\right]$ gives the equations $\mathbf{c}_B^T + \mathbf{u}^T\mathbf{A}_B = \mathbf{0}^T$ and $\mathbf{U}\mathbf{A}_B = \mathbf{I}$. We then get $\mathbf{u}^T = -\mathbf{c}_B^T\mathbf{A}_B^{-1}$ and $\mathbf{U} = \mathbf{A}_B^{-1}$, by solving the matrix equations. This proves the first part. The "In particular" statement, follows from block multiplication of matrices. □

The following theorem is essentially a proof that each step of the simplex method does what we want to to do. It also implies that if the LP has no degenerate basic feasible solutions that the simplex method terminates, because each LP only has a finite number of basic feasible solutions.

Theorem 9. *Let $\mathbf{x}$ be a bfs with associated basis B and $j_{in} \in \overline{B}$, and define $\mathbf{w} \in \mathbb{R}^n$ by $w_{j_{in}} = 1$, $\mathbf{w}_B = -\mathbf{A}_B^{-1}\mathbf{A}_{j_{in}}$, and $w_j = 0$ for all $j \notin B \cup \{j_{in}\}$. Furthermore, if $\mathbf{w}$ has a negative entry, then let*

$$\theta_0 = \min_{\substack{j \in B \\ w_j < 0}} \frac{x_j}{-w_j},$$

$J_{min} = \{j \in B : w_j < 0$ and $\frac{x_j}{-w_j} = \theta_0\}$ and $\mathbf{x}^* = \mathbf{x} + \theta_0\mathbf{w}$. The following statements are true:

(1) $\mathbf{w}$ is in the nullspace of A, (so $A(\mathbf{x} + \theta\mathbf{w}) = \mathbf{b}$ for any $\theta \in \mathbb{R}$) and $\mathbf{c}^T\mathbf{w} = \overline{c}_j$.
(2) If $\overline{c}_j < 0$ and $\mathbf{w} \geq \mathbf{0}$, then the LP is unbounded.

(3) If $\mathbf{w}$ has a negative entry, then $x_j^* = 0$ for all $j \in J_{\min}$, $\mathbf{c}^T\mathbf{x}^* = \mathbf{c}^T\mathbf{x} + \theta_0\overline{c}_j$, and $\mathbf{x}^*$ is a bfs with associated basis $B' = B \setminus \{j_{out}\} \cup \{j_{in}\}$ for any $j_{out} \in J_{min}$. In particular, if x is not degenerate, then $\mathbf{c}^T\mathbf{x}^* < \mathbf{c}^T\mathbf{x}$; and if $\mathbf{x}^*$ is not degenerate, then $|J_{min}| = 1$.

Proof. We have that

$$\mathbf{A}\mathbf{w} = \mathbf{A}_j + \mathbf{A}_B\mathbf{w}_B = \mathbf{A}_j + \mathbf{A}_B(-\mathbf{A}_B^{-1}\mathbf{A}_j) = \mathbf{0}$$

and

$$\mathbf{c}^T\mathbf{w} = c_j + \mathbf{c}_B^T\mathbf{w}_B = c_j + \mathbf{c}_B^T(-\mathbf{A}_B^{-1}\mathbf{A}_j) = c_j - \mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{A}_j = \overline{c}_j,$$

so (1) is true. Therefore, if $\mathbf{w} \geq 0$, then $\mathbf{x} + \theta\mathbf{w}$ is feasible for any $\theta \geq 0$, so when $\overline{c}_j < 0$, we have that

$$\mathbf{c}^T(\mathbf{x} + \theta\mathbf{w}) = \mathbf{c}^T\mathbf{x} + \theta\overline{c}_j \to -\infty$$

as θ goes to infinity, so (2) is true

Now assume $\mathbf{w}$ has a negative entry. Let $j \in [n]$. If $w_j \geq 0$, then clearly $x_j^* = x_j + \theta_0 w_j \geq 0$ and if $w_j < 0$, then

$$x_j^* = x_j + \theta_0 w_j \geq x_j + \frac{x_j}{-w_j}w_j = 0.$$

with equality holding if and only if $j \in J_{min}$. Clearly $\mathbf{c}^T\mathbf{x}^* = \mathbf{c}^T\mathbf{x} + \theta_0\mathbf{c}^T\mathbf{w} = \mathbf{c}^T\mathbf{x} + \theta_0\overline{c}_j$.

Let $j_{out} \in J_{min}$ and $B' = B \setminus \{j_{out}\} \cup \{j_{in}\}$. If $j \notin B'$, then $j \notin B$ or $j = j_{out}$ and in either case $\mathbf{x}_j^* = 0$. The only thing left to show is that B' is a basis. Let d_j be constants for each $j \in B'$ and assume that $\sum_{j\in B'} d_j\mathbf{A}_j = \mathbf{0}$. We will prove that $d_j = 0$ for all $j \in B'$, which will show that the columns of $\mathbf{A}'_B$ are linearly independent, i.e. $\mathbf{A}'_B$ is nonsingular, which will imply that B' is a basis. Recall that $\mathbf{A}_{j_{in}} = -\mathbf{A}_B w_B = \sum_{j\in B} -w_j\mathbf{A}_J$, hence

$$\begin{aligned}\mathbf{0} = \sum_{j\in B'} d_j\mathbf{A}_j &= d_{j_{in}}\mathbf{A}_{j_{in}} + \sum_{j\in B\setminus\{j_{out}\}} d_j\mathbf{A}_j = d_{j_{in}}\left(\sum_{j\in B} -w_j\mathbf{A}_j\right) + \sum_{j\in B\setminus\{j_{out}\}} d_j\mathbf{A}_j \\ &= -d_{j_{in}}w_{j_{out}}\mathbf{A}_{j_{out}} + \sum_{j\in B\setminus\{j_{out}\}} (-d_{j_{in}}w_j + d_j)\mathbf{A}_j,\end{aligned}$$

so since the columns of $\mathbf{A}_B$ are linearly independent, $-d_{j_{in}}w_{j_{out}} = 0$ and $-d_{j_{in}}w_j + d_j = 0$ for every $j \in B \setminus \{j_{out}\}$. Recall that $j_{out} \in J_{min}$ which implies that $w_{j_{out}} < 0$. Therefore, $d_{j_{in}} = 0$. This further implies that $d_j = 0$ for all $j \in B \setminus \{j_{out}\}$. Hence, we have shown that $d_j = 0$ for all $j \in B'$. □

References

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[2] C. Papadimitriou and K. Stieglitz, *Combinatorial Optimization*, Prentice Hall, 1982