# Matrix games

Any matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ defines a game, and we call $\mathbf{A}$ the *payout matrix*. The row player is *Alice* and each row is called a *pure strategy* for Alice. The column player is *Bob* and each column is a *pure strategy* for Bob. If Alice plays pure strategy $i$ and Bob play pure strategy $j$, Alice receives $a_{i,j}$ dollars from Bob. If $a_{i,j} < 0$, then this means that Alice pays Bob $-a_{i,j}$ dollars.

A vector $\mathbf{y}$ is *stochastic* if $\mathbf{y} \geq \mathbf{0}$ and $\sum_{i=1}^m y_i = 1$. Throughout, assume $\mathbf{x}, \mathbf{y}$ are stochastic vectors in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

Let $\mathbf{A} = \{a_{ij}\}$ be an $m \times n$ payoff matrix. Any $\mathbf{y}$ defines the *mixed strategy* for the row player, Alice, where she chooses each pure strategy $i$ with probability $y_i$, and any $\mathbf{x}$ defines a *mixed strategy* for the column player, Bob, where he chooses each pure strategy $j$ with probability $x_j$. If Alice plays mixed strategy $\mathbf{y}$ and Bob plays mixed strategy $\mathbf{x}$, then the *expected payout* is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} y_i a_{ij} x_j = \mathbf{y}^T A \mathbf{x}.$$

We say that stochastic $\tilde{\mathbf{y}} \in \mathbb{R}^m$ or stochastic $\tilde{\mathbf{x}} \in \mathbb{R}^n$ is *optimal* if it maximizes $\min_{\mathbf{x}} \tilde{\mathbf{y}}^T A \mathbf{x}$ or minimizes $\max_{\mathbf{y}} \mathbf{y}^T A \tilde{\mathbf{x}}$, respectively. It is not hard to see that for any stochastic $\bar{\mathbf{y}} \in \mathbb{R}^m$ and stochastic $\bar{\mathbf{x}} \in \mathbb{R}^n$,

$$\min_{\mathbf{x}} \bar{\mathbf{y}}^T A \mathbf{x} \leq \mathbf{y}^T A \mathbf{x} \leq \max_{\mathbf{y}} y^T A \bar{\mathbf{x}}. \tag{1}$$

**Theorem** (The Minimax Theorem)**.** *For every $A \in \mathbb{R}^{m \times n}$ there exists stochastic $\tilde{\mathbf{y}} \in \mathbb{R}^m$ and stochastic $\tilde{\mathbf{x}} \in \mathbb{R}^n$ such that*

$$\min_{\mathbf{x}} \tilde{\mathbf{y}}^T A \mathbf{x} = \tilde{\mathbf{y}}^T A \tilde{\mathbf{x}} = \max_{\mathbf{y}} \mathbf{y}^T A \tilde{\mathbf{x}}.$$

Note that, using (1), it can be shown that the mixed strategies $\tilde{\mathbf{y}}$ and $\tilde{\mathbf{x}}$ from the theorem are optimal. With $\tilde{\mathbf{y}}$ and $\tilde{\mathbf{x}}$ from the theorem, the number $\tilde{\mathbf{y}}^T A \tilde{\mathbf{x}}$ is called the *value* of the game. We first prove the following proposition ($\mathbf{e_i}$ is the $i$th standard basis vector).

**Proposition.** *For any $\mathbf{A} \in \mathbb{R}^{m \times n}$ and stochastic $\mathbf{y} \in \mathbb{R}^m$,*

$$\min_{\mathbf{x}} \mathbf{y}^T A \mathbf{x} = \min_{j} \mathbf{y}^T A \mathbf{e_j} = \min_{j} \sum_{i=1}^{m} y_i a_{i,j}.$$

*For any stochastic $\mathbf{x} \in \mathbb{R}^n$,*

$$\max_{\mathbf{y}} \mathbf{y}^T A \mathbf{x} = \max_{i} \mathbf{e_i}^T A \mathbf{x} = \max_{i} \sum_{j=1}^{n} a_{i,j} x_j.$$

*Proof.* Clearly, $\min_j \mathbf{y}^T A \mathbf{e_j} = \min_j \sum_{i=1}^m y_i a_{i,j}$ and $\min_{\mathbf{x}} \mathbf{y}^T A \mathbf{x} \leq \min_j \mathbf{y}^T A \mathbf{e_j}$ since every standard basis vector is stochastic. So we now only need to show $\min_{\mathbf{x}} \mathbf{y}^T A \mathbf{x} \geq \min_j \sum_{i=1}^m y_i a_{i,j}$ to prove the first sentence. Let $t = \min_j \sum_{i=1}^m y_i a_{i,j}$. For any stochastic $\mathbf{x} \in \mathbb{R}^n$,

$$\mathbf{y}^T A \mathbf{x} = \sum_{j=1}^{n} \sum_{i=1}^{m} y_i a_{i,j} x_j = \sum_{j=1}^{n} x_j \left( \sum_{i=1}^{m} y_i a_{i,j} \right) \geq \left( \sum_{j=1}^{n} x_j \right) \cdot t = t.$$

This implies that $\min_{\mathbf{x}} \mathbf{y}^T A \mathbf{x} \geq t$, which proves the first sentence. The proof of the second sentence is similar. $\square$

*Proof of the Minimax Theorem.* We want to find an optimal $\mathbf{x}$ which, by definition, is a stochastic vector $\mathbf{x}$ that minimizes $\max_{\mathbf{y}} \mathbf{y}^T \mathbf{A}\mathbf{x}$. By the proposition, this is equivalent finding a vector $\mathbf{x}$ that minimizes $\max_i \sum_{j=1}^m a_{i,j}x_j$ subject to $x_1 + \cdots + x_n = 1$ and $\mathbf{x} \geq \mathbf{0}$. By adding a variable $w$, we can write an LP whose solutions gives us the optimal $\mathbf{x}$ as follows,

$$
\begin{array}{lll}
\text{Minimize} & w & \\
\text{subject to} & x_1 + \cdots + x_n = 1 & \\
& w \geq \sum_{j=1}^n a_{i,j}x_j & \text{for all } i \in [n] \\
& \mathbf{x} \geq \mathbf{0} &
\end{array}
\tag{2}
$$

Similarly, we can write an LP whose solutions gives us an optimal $\mathbf{y}$ as follows

$$
\begin{array}{lll}
\text{Maximize} & u & \\
\text{subject to} & y_1 + \cdots + y_m = 1 & \\
& u \leq \sum_{i=1}^m a_{i,j}y_i & \text{for all } j \in [m] \\
& \mathbf{y} \geq \mathbf{0} &
\end{array}
\tag{3}
$$

It is a good exercise to show that (2) and (3) are dual linear programs, and this observation gives us the theorem. $\square$