Math-484 Homework #3 (semidefinite matrices, convex sets and convex functions)

Due 11am Sep 24.
Write your name on your solutions and indicate if you are a D14 (4 credit hour) student.

1: Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = x^3 + e^{3y} - 3xe^y.$$

Show that f has exactly one critical point and that this point is a local minimizer but not a global minimizer.

2: (a) Decide if the following matrix is (positive/negative,semi)definite:

$$A = \begin{pmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

(b) Decide for which $t \in \mathbb{R}$ is the following matrix is positive definite:

$$B = \begin{pmatrix} t & 1 & 0 \\ 1 & t & 1 \\ 0 & 1 & t \end{pmatrix}$$

3: Let $\mathcal{PS}$ be the set of all positive semidefinite definite $n \times n$ matrices in $\mathbb{R}^{n\times n}$. Use the definition of convexity to show that $\mathcal{PS}$ is a convex set.

4: For $D \subseteq \mathbb{R}^n$ we define $co(D)$ to be the intersection of all convex sets containing D. Prove Theorem 2.1.4: Let $D \subseteq \mathbb{R}^n$. Then $co(D)$ coincides with the set C of all convex combinations of vectors from D.
Hint: Use the following steps.
1) Show that C is a convex set containing D.
2) Show that if B is a convex set containing D, then $C \subseteq B$ (Use a theorem for this step).
3) Conclude that $co(D) = C$.

5: Determine whether the functions are convex, concave, strictly convex or strictly concave on the specified sets:
(a) $f(x) = \ln x$ for $x \in (0, +\infty)$
(b) $f(x_1, x_2) = 5x_1^2 + 2x_1x_2 + x_2^2 - x_1 + 2x_2 + 3$ for $(x_1, x_2) \in \mathbb{R}^2$
(c) $f(x_1, x_2) = (x_1 + 2x_2 + 1)^8 - \ln((x_1x_2)^2)$ for $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > x_2 > 1\}$
(d) $f(x_1, x_2) = c_1x_1 + c_2/x_1 + c_3x_2 + c_4/x_2$ for $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$, where c_1, c_2, c_3, and c_4 are positive constants

6: *D14 only* Suppose that f is a function with continuous second partial derivatives on $\mathbb{R}^n$. Show that if there exists x^* such that the Hessian $Hf(x^*)$ of f at x^* is not positive semidefinite, then f is not convex.