Buying a car usually requires both some savings for a down payment and a loan for the balance. An exercise in Section 2 calculates the regular deposits that would be needed to save up the full purchase price, and other exercises and examples in this chapter compute the payments required to amortize a loan.
Whether you are in a position to invest money or to borrow money, it is important for both consumers and business managers to understand interest. The formulas for interest are developed in this chapter.

5.1 SIMPLE AND COMPOUND INTEREST

**THINK ABOUT IT**

If you can borrow money at 11% interest compounded annually or at 10.8% compounded monthly, which loan would cost less?

We shall see how to make such comparisons in this section.

**Simple Interest**  Interest on loans of a year or less is frequently calculated as simple interest, a type of interest that is charged (or paid) only on the amount borrowed (or invested), and not on past interest. The amount borrowed is called the principal. The rate of interest is given as a percent per year, expressed as a decimal. For example, $6\% = .06$ and $11\frac{1}{2}\% = .115$. The time the money is earning interest is calculated in years. Simple interest is the product of the principal, rate, and time.

**SIMPLE INTEREST**

\[ I = Prt, \]

where \( p \) is the principal;
\( r \) is the annual interest rate;
\( t \) is the time in years.

**EXAMPLE 1**  Simple Interest

To buy furniture for a new apartment, Jennifer Wall borrowed $5000 at 8% simple interest for 11 months. How much interest will she pay?

**Solution**  From the formula, \( I = Prt \), with \( P = 5000 \), \( r = .08 \), and \( t = 11/12 \) (in years). The total interest she will pay is

\[ I = 5000(.08)(11/12) = 366.67, \]

or $366.67.

A deposit of \( P \) dollars today at a rate of interest \( r \) for \( t \) years produces interest of \( I = Prt \). The interest, added to the original principal \( P \), gives

\[ P + Prt = P(1 + rt). \]

This amount is called the future value of \( P \) dollars at an interest rate \( r \) for time \( t \) in years. When loans are involved, the future value is often called the maturity value of the loan. This idea is summarized as follows.
5.1 Simple and Compound Interest

FUTURE OR MATURITY VALUE FOR SIMPLE INTEREST

The future or maturity value \( A \) of \( P \) dollars at a simple interest rate \( r \) for \( t \) years is

\[
A = P(1 + rt).
\]

EXAMPLE 2

Maturity Values

Find the maturity value for each loan at simple interest.

(a) A loan of $2500 to be repaid in 8 months with interest of 9.2%.

Solution

The loan is for 8 months, or \( \frac{8}{12} = \frac{2}{3} \) of a year. The maturity value is

\[
A = P(1 + rt)
\]

\[
A = 2500 \left[ 1 + \frac{0.092}{3} \right]
\]

\[
A \approx 2500(1 + 0.06133) \approx 2653.33,
\]

or $2653.33. (The answer is rounded to the nearest cent, as is customary in financial problems.) Of this maturity value,

\[
2653.33 - 2500 = 153.33
\]

represents interest.

(b) A loan of $11,280 for 85 days at 11% interest.

Solution

It is common to assume 360 days in a year when working with simple interest. We shall usually make such an assumption in this book. The maturity value in this example is

\[
A = P(1 + rt)
\]

\[
A = 11,280 \left[ 1 + \frac{0.11 \times 85}{360} \right]
\]

\[
A = 11,572.97,
\]

or $11,572.97.

CAUTION

When using the formula for future value, as well as all other formulas in this chapter, we neglect the fact that in real life, money amounts are rounded to the nearest penny. As a consequence, when the amounts are rounded, their values may differ by a few cents from the amounts given by these formulas. For instance, in Example 2(a), the interest in each monthly payment would be $2500(0.092/12) \approx 19.17, rounded to the nearest penny. After 8 months, the total is 8($19.17) = 153.36, which is 3¢ more than we computed in the example.

In part (b) of Example 2 we assumed 360 days in a year. Interest found using a 360-day year is called ordinary interest, and interest found using a 365-day year is called exact interest.

The formula for future value has four variables, \( P, r, t, \) and \( A \). We can use the formula to find any of the quantities that these variables represent, as illustrated in the next example.
Chapter 5  Mathematics of Finance

**Simple Interest**

Carter Fenton wants to borrow $8000 from Christine O’Brien. He is willing to pay back $8380 in 6 months. What interest rate will he pay?

**Solution** Use the formula for future value, with $A = 8380$, $P = 8000$, $t = 6/12 = .5$, and solve for $r$.

\[
A = P(1 + rt)
\]

\[
8380 = 8000(1 + .5r)
\]

\[
8380 = 8000 + 4000r\quad\text{Distributive property}
\]

\[
380 = 4000r \quad\text{Subtract 8000.}
\]

\[
r = .095 \quad\text{Divide by 4000.}
\]

Thus, the interest rate is 9.5%.

**Compound Interest** As mentioned earlier, simple interest is normally used for loans or investments of a year or less. For longer periods compound interest is used. With compound interest, interest is charged (or paid) on interest as well as on principal. For example, if $1000 is deposited at 5% interest for 1 year, at the end of the year the interest is $1000(.05)(1) = $50. The balance in the account is $1000 + $50 = $1050. If this amount is left at 5% interest for another year, the interest is calculated on $1050 instead of the original $1000, so the amount in the account at the end of the second year is $1050 + $1050(.05)(1) = $1102.50. Note that simple interest would produce a total amount of only

\[
$1000[1 + (.05)(2)] = $1100.
\]

To find a formula for compound interest, first suppose that $P$ dollars is deposited at a rate of interest $r$ per year. The amount on deposit at the end of the first year is found by the simple interest formula, with $t = 1$.

\[
A = P(1 + r \cdot 1) = P(1 + r)
\]

If the deposit earns compound interest, the interest earned during the second year is paid on the total amount on deposit at the end of the first year. Using the formula $A = P(1 + rt)$ again, with $P$ replaced by $P(1 + r)$ and $t = 1$, gives the total amount on deposit at the end of the second year.

\[
A = [P(1 + r)](1 + r \cdot 1) = P(1 + r)^2
\]

In the same way, the total amount on deposit at the end of the third year is

\[
P(1 + r)^3.
\]

Generalizing, in $t$ years the total amount on deposit is

\[
A = P(1 + r)^t,
\]

called the compound amount.
Note: Compare this formula for compound interest with the formula for simple interest.

### Compound Interest

\[
A = P(1 + r)^t
\]

### Simple Interest

\[
A = P(1 + rt)
\]

The important distinction between the two formulas is that in the compound interest formula, the number of years, \( t \), is an exponent, so that money grows much more rapidly when interest is compounded.

Interest can be compounded more than once per year. Common compounding periods include *semiannually* (two periods per year), *quarterly* (four periods per year), *monthly* (twelve periods per year), or *daily* (usually 365 periods per year). The interest rate per period, \( i \), is found by dividing the annual interest rate, \( r \), by the number of compounding periods, \( m \), per year. To find the total number of compounding periods, \( n \), we multiply the number of years, \( t \), by the number of compounding periods per year, \( m \). The following formula can be derived in the same way as the previous formula.

### Compound Amount

\[
A = P(1 + i)^n,
\]

where \( i = \frac{r}{m} \) and \( n = mt \).

- **\( A \)** is the future (maturity) value;
- **\( P \)** is the principal;
- **\( r \)** is the annual interest rate;
- **\( m \)** is the number of compounding periods per year;
- **\( t \)** is the number of years;
- **\( n \)** is the number of compounding periods;
- **\( i \)** is the interest rate per period.

#### Example 4

**Compound Interest**

Suppose $1000 is deposited for 6 years in an account paying 4.25% per year compounded annually.

(a) Find the compound amount.

**Solution** In the formula above, \( P = 1000, i = .0425/1, \) and \( n = 6(1) = 6 \).

The compound amount is

\[
A = P(1 + i)^n
\]

\[
A = 1000(1.0425)^6.
\]

Using a calculator, we get

\[
A = \$1283.68,
\]

the compound amount.
(b) Find the amount of interest earned.

Solution Subtract the initial deposit from the compound amount.

Amount of interest = $1283.68 - $1000 = $283.68

**EXAMPLE 5** Compound Interest

Find the amount of interest earned by a deposit of $2450 for 6.5 years at 5.25% compounded quarterly.

**Solution** Interest compounded quarterly is compounded 4 times a year. In 6.5 years, there are 6.5(4) = 26 periods. Thus, \(n = 26\). Interest of 5.25% per year is 5.25%/4 per quarter, so \(i = .0525/4\). Now use the formula for compound amount.

\[
A = P(1 + i)^n
\]

\[
A = 2450(1 + .0525/4)^{26} = 3438.78
\]

Rounded to the nearest cent, the compound amount is $3438.78, so the interest is $3438.78 - $2450 = $988.78.

**CAUTION** As shown in Example 5, compound interest problems involve two rates—the annual rate \(r\) and the rate per compounding period \(i\). Be sure you understand the distinction between them. When interest is compounded annually, these rates are the same. In all other cases, \(i \neq r\).

It is interesting to compare loans at the same rate when simple or compound interest is used. Figure 1 shows the graphs of the simple interest and compound interest formulas with \(P = 1000\) at an annual rate of 10% from 0 to 20 years. The future value after 15 years is shown for each graph. After 15 years at compound interest, $1000 grows to $4177.25, whereas with simple interest, it amounts to $2500.00, a difference of $1677.25.

Spreadsheets are ideal for performing financial calculations. Figure 2 (on the next page) shows a Microsoft Excel spreadsheet with the formulas for compound and simple interest used to create columns B and C, respectively, when $1000 is invested at an annual rate of 10%. Compare row 16 with the calculator results in Figure 1. For more details on the use of spreadsheets in the mathematics of finance, see *The Spreadsheet Manual* that is available with this book.

**Effective Rate** If $1 is deposited at 4% compounded quarterly, a calculator can be used to find that at the end of one year, the compound amount is $1.0406, an increase of 4.06% over the original $1. The actual increase of 4.06% in the
5.1 Simple and Compound Interest

When applied to consumer finance, the effective rate is called the annual percentage rate, APR, or annual percentage yield, APY. Money is somewhat higher than the stated increase of 4%. To differentiate between these two numbers, 4% is called the nominal or stated rate of interest, while 4.06% is called the effective rate. To avoid confusion between stated rates and effective rates, we shall continue to use \( r \) for the stated rate and we will use \( r_e \) for the effective rate.

**EXAMPLE 6**

**Effective Rate**

Find the effective rate corresponding to a stated rate of 6% compounded semiannually.

**Solution** Here, \( r/m = 6%/2 = 3% \) for \( m = 2 \) periods. Use a calculator to find that \((1.03)^2 = 1.06090\), which shows that $1 will increase to $1.06090, an actual increase of 6.09%. The effective rate is \( r_e = 6.09\% \).

Generalizing from this example, the effective rate of interest is given by the following formula.

**EFFECTIVE RATE**

The effective rate corresponding to a stated rate of interest \( r \) compounded \( m \) times per year is

\[
 r_e = \left(1 + \frac{r}{m}\right)^m - 1.
\]

*When applied to consumer finance, the effective rate is called the annual percentage rate, APR, or annual percentage yield, APY.*
EXAMPLE 7  Effective Rate
A bank pays interest of 4.9% compounded monthly. Find the effective rate.

Solution  Use the formula given above with \( r = 0.049 \) and \( m = 12 \). The effective rate is

\[
re = \left(1 + \frac{0.049}{12}\right)^{12} - 1 = 0.05011575,
\]

or 5.01%.

EXAMPLE 8  Effective Rate
Joe Vetere needs to borrow money. His neighborhood bank charges 11% interest compounded semiannually. A downtown bank charges 10.8% interest compounded monthly. At which bank will Joe pay the lesser amount of interest?

Solution  Compare the effective rates.

Neighborhood bank:  \[
re = \left(1 + \frac{0.11}{2}\right)^{2} - 1 = 0.113025 \approx 11.3\% 
\]

Downtown bank:  \[
re = \left(1 + \frac{0.108}{12}\right)^{12} - 1 \approx 0.11351 \approx 11.4\%
\]

The neighborhood bank has the lower effective rate, although it has a higher stated rate.

Present Value  The formula for compound interest, \( A = P(1 + i)^n \), has four variables: \( A \), \( P \), \( i \), and \( n \). Given the values of any three of these variables, the value of the fourth can be found. In particular, if \( A \) (the future amount), \( i \), and \( n \) are known, then \( P \) can be found. Here \( P \) is the amount that should be deposited today to produce \( A \) dollars in \( n \) periods.

EXAMPLE 9  Present Value
Rachel Reeve must pay a lump sum of $6000 in 5 years. What amount deposited today at 6.2% compounded annually will amount to $6000 in 5 years?

Solution  Here \( A = 6000 \), \( i = 0.062 \), \( n = 5 \), and \( P \) is unknown. Substituting these values into the formula for the compound amount gives

\[
6000 = P(1.062)^5
\]

\[
P = \frac{6000}{(1.062)^5} \approx 4441.49,
\]

or $4441.49. If Rachel leaves $4441.49 for 5 years in an account paying 6.2% compounded annually, she will have $6000 when she needs it. To check your work, use the compound interest formula with \( P = 4441.49 \), \( i = 0.062 \), and \( n = 5 \). You should get \( A = 6000.00 \).
As Example 9 shows, $6000 in 5 years is approximately the same as $4441.49 today (if money can be deposited at 6.2% compounded annually). An amount that can be deposited today to yield a given sum in the future is called the present value of the future sum. Generalizing from Example 9, by solving \( A = P(1 + i)^n \) for \( P \), we get the following formula for present value.

**PRESENT VALUE FOR COMPOUND INTEREST**

The present value of \( A \) dollars compounded at an interest rate \( i \) per period for \( n \) periods is

\[
P = \frac{A}{(1 + i)^n} \quad \text{or} \quad P = A(1 + i)^{-n}.
\]

**EXAMPLE 10** Present Value

Find the present value of $16,000 in 9 years if money can be deposited at 6% compounded semiannually.

**Solution** In 9 years there are \( 2 \cdot 9 = 18 \) semiannual periods. A rate of 6% per year is 3% in each semiannual period. Apply the formula with \( A = 16,000 \), \( i = .03 \), and \( n = 18 \).

\[
P = \frac{A}{(1 + i)^n} = \frac{16,000}{(1.03)^{18}} \approx 9398.31
\]

A deposit of $9398.31 today, at 6% compounded semiannually, will produce a total of $16,000 in 9 years.

We can solve the compound amount formula for \( n \) also, as the following example shows.

**EXAMPLE 11** Price Doubling

Suppose the general level of inflation in the economy averages 8% per year. Find the number of years it would take for the overall level of prices to double.

**Solution** To find the number of years it will take for $1 worth of goods or services to cost $2, find \( n \) in the equation

\[
2 = 1(1 + .08)^n,
\]

where \( A = 2 \), \( P = 1 \), and \( i = .08 \). This equation simplifies to

\[
2 = (1.08)^n.
\]

By trying various values of \( n \), we find that \( n = 9 \) is approximately correct, because \( 1.08^9 = 1.99900 \approx 2 \). The exact value of \( n \) can be found quickly by using logarithms, but that is beyond the scope of this chapter. Thus, the overall level of prices will double in about 9 years.
At this point, it seems helpful to summarize the notation and the most important formulas for simple and compound interest. We use the following variables.

\[ P \] = principal or present value
\[ A \] = future or maturity value
\[ r \] = annual (stated or nominal) interest rate
\[ t \] = number of years
\[ m \] = number of compounding periods per year
\[ i \] = interest rate per period \[ i = \frac{r}{m} \]
\[ n \] = total number of compounding periods \[ n = tm \]
\[ r_e \] = effective rate

### Simple Interest

\[ A = P(1 + rt) \]
\[ P = \frac{A}{1 + rt} \]

### Compound Interest

\[ A = P(1 + i)^n \]
\[ P = \frac{A}{(1 + i)^n} = A(1 + i)^{-n} \]
\[ r_e = \left(1 + \frac{r}{m}\right)^m - 1 \]

## 5.1 Exercises

1. What is the difference between \( r \) and \( i \)? between \( t \) and \( n \)?
2. We calculated the loan in Example 2(b) assuming 360 days in a year. Find the maturity value using 365 days in a year. Which is more advantageous to the borrower?
3. What factors determine the amount of interest earned on a fixed principal?
4. In your own words, describe the maturity value of a loan.
5. What is meant by the present value of money?

**Find the simple interest.**

6. $25,000 at 7% for 9 months
7. $3850 at 9% for 8 months
8. $1974 at 6.3% for 7 months
9. $3724 at 8.4% for 11 months

**Find the simple interest. Assume a 360-day year.**

10. $5147.18 at 10.1% for 58 days
11. $2930.42 at 11.9% for 123 days

12. Explain the difference between simple interest and compound interest.

13. In Figure 1, one graph is a straight line and the other is curved. Explain why this is, and which represents each type of interest.

**Find the compound amount for each deposit.**

14. $1000 at 6% compounded annually for 8 years
15. $1000 at 7% compounded annually for 10 years
16. $470 at 10% compounded semiannually for 12 years
17. $15,000 at 6% compounded semiannually for 11 years
18. $6500 at 12% compounded quarterly for 6 years
19. $9100 at 8% compounded quarterly for 4 years

**Find the amount that should be invested now to accumulate the following amounts, if the money is compounded as indicated.**

20. $15,902.74 at 9.8% compounded annually for 7 years
21. $27,159.68 at 12.3% compounded annually for 11 years
5.1 Simple and Compound Interest

22. $2000 at 9% compounded semiannually for 8 years
23. $2000 at 11% compounded semiannually for 8 years
24. $8800 at 10% compounded quarterly for 5 years
25. $7500 at 12% compounded quarterly for 9 years
26. How do the nominal or stated interest rate and the effective interest rate differ?
27. If interest is compounded more than once per year, which rate is higher, the stated rate or the effective rate?

Find the effective rate corresponding to each nominal rate.

28. 3% compounded quarterly
29. 8% compounded quarterly
30. 8.25% compounded semiannually
31. 10.08% compounded semiannually

Applications

BUSINESS AND ECONOMICS

32. Loan Repayment Susan Carsten borrowed $25,900 from her father to start a flower shop. She repaid him after 11 mo, with interest of 8.4%. Find the total amount she repaid.

33. Delinquent Taxes An accountant for a corporation forgot to pay the firm’s income tax of $725,896.15 on time. The government charged a penalty of 12.7% interest for the 34 days the money was late. Find the total amount (tax and penalty) that was paid. (Use a 365-day year.)

34. Savings A $100,000 certificate of deposit held for 60 days was worth $101,133.33. To the nearest tenth of a percent, what interest rate was earned?

35. Savings A firm of accountants has ordered 7 new IBM computers at a cost of $5104 each. The machines will not be delivered for 7 months. What amount could the firm deposit in an account paying 6.42% to have enough to pay for the machines?

36. Stock Growth A stock that sold for $22 at the beginning of the year was selling for $24 at the end of the year. If the stock paid a dividend of $.50 per share, what is the simple interest rate on an investment in this stock? (Hint: Consider the interest to be the increase in value plus the dividend.)

37. Bond Interest A bond with a face value of $10,000 in 10 years can be purchased now for $5988.02. What is the simple interest rate?

38. Loan Interest A small business borrows $50,000 for expansion at 12% compounded monthly. The loan is due in 4 years. How much interest will the business pay?

39. Wealth A 1997 article in The New York Times discussed how long it would take for Bill Gates, the world’s second richest person at the time (behind the Sultan of Brunei), to become the world’s first trillionaire.* His birthday is October 28, 1955, and on July 16, 1997, he was worth $42 billion. (Note: A trillion dollars is 1000 billion dollars.)

a. Assume that Bill Gates’s fortune grows at an annual rate of 58%, the historical growth rate through 1997 of Microsoft stock, which made up most of his wealth in 1997. Find the age at which he becomes a trillionaire.

b. Repeat part a using 10.9% growth, the average return on all stocks since 1926.

c. What rate of growth would be necessary for Bill Gates to become a trillionaire by the time he is eligible for Social Security on January 1, 2022, after he has turned 66?

d. An article on September 19, 1999, gave Bill Gates’s wealth as roughly $90 billion. What was the rate of growth of his wealth between the 1997 and 1999 articles?†

43. **Comparing Investments** Two partners agree to invest equal amounts in their business. One will contribute $10,000 immediately. The other plans to contribute an equivalent amount in 3 years, when she expects to acquire a large sum of money. How much should she contribute at that time to match her partner’s investment now, assuming an interest rate of 6% compounded semiannually?

44. **Comparing Investments** As the prize in a contest, you are offered $1000 now or $1210 in 5 years. If money can be invested at 6% compounded annually, which is larger?

45. **Comparing CD Rates** A Virginia bank offered the following special on CD (certificate of deposit) rates. The rates are annual percentage yields, or effective rates, which are higher than the corresponding nominal rates. Assume quarterly compounding. Solve for $r$ to approximate the corresponding nominal rates to the nearest hundredth.

<table>
<thead>
<tr>
<th>Term</th>
<th>6 mo</th>
<th>1 yr</th>
<th>18 mo</th>
<th>2 yr</th>
<th>3 yr</th>
</tr>
</thead>
<tbody>
<tr>
<td>APY(%)</td>
<td>1.30</td>
<td>2.17</td>
<td>2.27</td>
<td>2.55</td>
<td>3.00</td>
</tr>
</tbody>
</table>

46. **Effective Rate** An advertisement for E*TRADE Bank boasted “We’re ahead of banks that had a 160-year start,” with an APY (or effective rate) of 2.01%. The actual rate was not stated. Given that interest was compounded monthly, find the actual rate.

47. **Effective Rate** According to a financial Web site, on July 23, 2003, Countrywide Bank of Alexandria, Virginia, paid 2.03% interest, compounded daily, on a 1-year CD, while New South Federal Savings of Birmingham, Alabama, paid 2.05% compounded semiannually. What are the effective rates for the two CDs, and which bank pays a higher effective rate?

48. **Retirement Savings** The pie graph below shows the percent of baby boomers aged 46–49 who said they had investments with a total value as shown in each category.

- Don’t know or no answer: 28%
- Less than $10,000: 30%
- $10,000 to $1 million: 13%
- $150,000 to $1 million: 29%
- More than $1 million: 5%

Figures add to more than 100% because of rounding.

Note that 30% have saved less than $10,000. Assume the money is invested at an average rate of 8% compounded quarterly. What will the top numbers in each category amount to in 20 years, when this age group will be ready for retirement?

**Doubling Time** Use the ideas from Example 11 to find the time it would take for the general level of prices in the economy to double at each average annual inflation rate.

49. 4% 50. 5%

51. **Doubling Time** The consumption of electricity has increased historically at 6% per year. If it continues to increase at this rate indefinitely, find the number of years before the electric utilities will need to double their generating capacity.

52. **Doubling Time** Suppose a conservation campaign coupled with higher rates causes the demand for electricity to increase at only 2% per year, as it has recently. Find the number of years before the utilities will need to double generating capacity.

**Negative Interest** Under certain conditions, Swiss banks pay negative interest: they charge you. (You didn’t think all that secrecy was free?) Suppose a bank “pays” –2.4% interest compounded annually. Find the compound amount for a deposit of $150,000 after each period.

53. 4 years 54. 8 years

55. **Interest Rate** In 1995, O. G. McClain of Houston, Texas, mailed a $100 check to a descendant of Texas independence hero Sam Houston to repay a $100 debt of McClain’s great-great-grandfather, who died in 1835, to Sam Houston. A bank estimated the interest on the loan to be $420 million for the 160 years it was due. Find the interest rate the bank was using, assuming interest is compounded annually.

56. **Investment** In the New Testament, Jesus commends a widow who contributed 2 mites to the temple treasury (Mark 12:42–44). A mite was worth roughly 1/8 of a cent. Suppose the temple invested those 2 mites at 4% interest compounded quarterly. How much would the money be worth 2000 years later?

57. **Investments** Sun Kang borrowed $5200 from his friend Hop Fong Yee to pay for remodeling work on his house. He repaid the loan 10 months later with simple interest at 7%. Yee then invested the proceeds in a 5-year certificate of deposit paying 6.3% compounded quarterly. How much will...
5.2 Future Value of an Annuity

58. **Investments** Suppose $10,000 is invested at an annual rate of 5% for 10 years. Find the future value if interest is compounded as follows.
   a. Annually
   b. Quarterly
   c. Monthly
   d. Daily (365 days)

59. **Investments** In Exercise 58, notice that as the money is compounded more often, the compound amount becomes larger and larger. Is it possible to compound often enough so that the compound amount is $17,000 after 10 years? Explain.

---

5.2 FUTURE VALUE OF AN ANNUITY

**THINK ABOUT IT**

If you deposit $1500 each year for 6 years in an account paying 8% interest compounded annually, how much will be in your account at the end of this period?

In this section and the next, we develop future value and present value formulas for such periodic payments. To develop these formulas, we must first discuss sequences.

**Geometric Sequences** If $a$ and $r$ are nonzero real numbers, the infinite list of numbers $a, ar, ar^2, ar^3, \ldots , ar^n, \ldots$ is called a *geometric sequence*. For example, if $a = 3$ and $r = -2$, we have the sequence

\[3, 3(-2), 3(-2)^2, 3(-2)^3, 3(-2)^4, \ldots,\]

or

\[3, -6, 12, -24, 48, \ldots.\]

In the sequence $a, ar, ar^2, ar^3, ar^4, \ldots$, the number $a$ is called the first term of the sequence, $ar$ is the second term, $ar^2$ is the third term, and so on. Thus, for any $n \geq 1$,

\[ar^{n-1}\text{ is the } n\text{th term of the sequence.}\]

Each term in the sequence is $r$ times the preceding term. The number $r$ is called the *common ratio* of the sequence.

**EXAMPLE 1** Geometric Sequence

Find the seventh term of the geometric sequence 5, 20, 80, 320, \ldots.

---

*Problem 5 from “Course 140 Examination, Mathematics of Compound Interest” of the Education and Examination Committee of The Society of Actuaries. Reprinted by permission of The Society of Actuaries.*
Solution  Here, $a = 5$ and $r = 20/5 = 4$. We want the seventh term, so $n = 7$. Use $ar^{n-1}$, with $a = 5$, $r = 4$, and $n = 7$.

$$ar^{n-1} = (5)(4)^{7-1} = 5(4)^6 = 20,480$$

**EXAMPLE 2**  Geometric Sequence

Find the first five terms of the geometric sequence with $a = 10$ and $r = 2$.

**Solution**  The first five terms are

$$10, 10(2), 10(2)^2, 10(2)^3, 10(2)^4,$$

or

$$10, 20, 40, 80, 160.$$  

Next, we need to find the sum $S_n$ of the first $n$ terms of a geometric sequence, where

$$S_n = a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1}. \quad (1)$$

If $r = 1$, then

$$S_n = a + a + a + a + \cdots + a = na.$$  

If $r \neq 1$, multiply both sides of equation (1) by $r$ to get

$$rS_n = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^n. \quad (2)$$

Now subtract corresponding sides of equation (1) from equation (2).

$$rS_n - S_n = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + ar^n - (a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1})$$

$$rS_n - S_n = -a + ar^n$$

$$S_n(r - 1) = a(r^n - 1) \quad \text{Factor}$$

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad \text{Divide both sides by } r - 1.$$  

This result is summarized below.

**SUM OF TERMS**

If a geometric sequence has first term $a$ and common ratio $r$, then the sum $S_n$ of the first $n$ terms is given by

$$S_n = \frac{a(r^n - 1)}{r - 1}, \quad r \neq 1.$$  

**EXAMPLE 3**  Sum of a Geometric Sequence

Find the sum of the first six terms of the geometric sequence 3, 12, 48, ...
5.2 Future Value of an Annuity

Solution

Here \( a = 3 \) and \( r = 4 \). Find \( S_6 \) by the formula above.

\[
S_6 = \frac{3(4^6 - 1)}{4 - 1} = \frac{3(4096 - 1)}{3} = 4095
\]

Ordinary Annuities

A sequence of equal payments made at equal periods of time is called an **annuity**. If the payments are made at the end of the time period, and if the frequency of payments is the same as the frequency of compounding, the annuity is called an **ordinary annuity**. The time between payments is the **payment period**, and the time from the beginning of the first payment period to the end of the last period is called the **term** of the annuity. The **future value of the annuity**, the final sum on deposit, is defined as the sum of the compound amounts of all the payments, compounded to the end of the term.

Two common uses of annuities are to accumulate funds for some goal or to withdraw funds from an account. For example, an annuity may be used to save money for a large purchase, such as an automobile, an expensive trip, or a down payment on a home. An annuity also may be used to provide monthly payments for retirement. We explore these options in this and the next section.

For example, suppose $1500 is deposited at the end of each year for the next 6 years in an account paying 8% per year compounded annually. Figure 3 shows this annuity. To find the future value of the annuity, look separately at each of the $1500 payments. The first of these payments will produce a compound amount of

\[
1500(1 + .08)^5 = 1500(1.08)^5.
\]

Use 5 as the exponent instead of 6 since the money is deposited at the end of the year and earns interest for only 5 years. The second payment of $1500 will produce a compound amount of $1500(1.08)^4$. As shown in Figure 4 on the next page, the future value of the annuity is

\[
1500(1.08)^5 + 1500(1.08)^4 + 1500(1.08)^3 + 1500(1.08)^2 + 1500(1.08)^1 + 1500.
\]

(The last payment earns no interest at all.)
Reading this sum in reverse order, we see that it is the sum of the first six terms of a geometric sequence, with \( a = 1500 \), \( r = 1.08 \), and \( n = 6 \). Thus, the sum equals

\[
\frac{a(r^n - 1)}{r - 1} = \frac{1500[(1.08)^6 - 1]}{1.08 - 1} \approx 11,003.89.
\]

To generalize this result, suppose that payments of \( R \) dollars each are deposited into an account at the end of each period for \( n \) periods, at a rate of interest \( i \) per period. The first payment of \( R \) dollars will produce a compound amount of \( R(1 + i)^{n-1} \) dollars, the second payment will produce \( R(1 + i)^{n-2} \) dollars, and so on; the final payment earns no interest and contributes just \( R \) dollars to the total. If \( S \) represents the future value (or sum) of the annuity, then (as shown in Figure 5 below),

\[
S = R(1 + i)^{n-1} + R(1 + i)^{n-2} + R(1 + i)^{n-3} + \cdots + R(1 + i) + R,
\]

or, written in reverse order,

\[
S = R + R(1 + i) + R(1 + i)^2 + \cdots + R(1 + i)^{n-1}.
\]

This result is the sum of the first \( n \) terms of the geometric sequence having first term \( R \) and common ratio \( 1 + i \). Using the formula for the sum of the first \( n \) terms of a geometric sequence,

\[
S = R \left( \frac{(1 + i)^n - 1}{i} \right) = R \left[ \frac{(1 + i)^n - 1}{i} \right].
\]

The quantity in brackets is commonly written \( s_{\overline{n|}} \) (read “\( s \)-angle-\( n \) at \( i \)”), so that

\[
S = R \cdot s_{\overline{n|}},
\]
5.2 Future Value of an Annuity

Values of $s_{n}$ can be found with a calculator.

A formula for the future value of an annuity $S$ of $n$ payments of $R$ dollars each at the end of each consecutive interest period, with interest compounded at a rate $i$ per period, follows.* Recall that this type of annuity, with payments at the end of each time period, is called an ordinary annuity.

**FUTURE VALUE OF AN ORDINARY ANNUITY**

$$S = R \left[ \frac{(1 + i)^n - 1}{i} \right] \quad \text{or} \quad S = Rs \frac{i}{1 - i}$$

where
- $S$ is the future value;
- $R$ is the payment;
- $i$ is the interest rate per period;
- $n$ is the number of periods.

A calculator will be very helpful in computations with annuities. The TI-83/84 Plus graphing calculator has a special FINANCE menu that is designed to give any desired result after entering the basic information. If your calculator does not have this feature, many calculators can easily be programmed to evaluate the formulas introduced in this section and the next. We include these programs in *The Graphing Calculator Manual* available for this text.

**EXAMPLE 4**

Ordinary Annuity

Karen Scott is an athlete who believes that her playing career will last 7 years. To prepare for her future, she deposits $22,000 at the end of each year for 7 years in an account paying 6% compounded annually. How much will she have on deposit after 7 years?

**Solution** Her payments form an ordinary annuity, with $r = 22,000$, $n = 7$, and $i = .06$. The future value of this annuity (by the formula above) is

$$S = 22,000 \left( \frac{(1.06)^7 - 1}{.06} \right) \approx 184,664.43,$$

or $184,664.43$.

**Sinking Funds** A fund set up to receive periodic payments as in Example 4 is called a sinking fund. The periodic payments, together with the interest earned by the payments, are designed to produce a given sum at some time in the future. For example, a sinking fund might be set up to receive money that will be needed to pay off the principal on a loan at some future time. If the payments are all the same amount and are made at the end of a regular time period, they form an ordinary annuity.

*We use $S$ for the future value here, instead of $A$ as in the compound interest formula, to help avoid confusing the two formulas.
EXAMPLE 5  Sinking Fund

Experts say that the baby boom generation (Americans born between 1946 and 1960) cannot count on a company pension or Social Security to provide a comfortable retirement, as their parents did. It is recommended that they start to save early and regularly. Sarah Santora, a baby boomer, has decided to deposit $200 each month for 20 years in an account that pays interest of 7.2% compounded monthly.

(a) How much will be in the account at the end of 20 years?

Solution  This savings plan is an annuity with 

\[
S = 200 \left( \frac{(1 + \frac{.072}{12})^{12(20)} - 1}{\frac{.072}{12}} \right) \approx 106,752.47,
\]

or $106,752.47. Figure 6 shows a calculator graph of the function

\[
S = 200 \left( \frac{(1 + \frac{x}{12})^{12(20)} - 1}{\frac{x}{12}} \right)
\]

where \( r \), the annual interest rate, is designated \( x \). The value of the function at \( x = .072 \), shown at the bottom of the window, agrees with our result above.

(b) Sarah believes she needs to accumulate $130,000 in the 20-year period to have enough for retirement. What interest rate would provide that amount?

Solution

Method 1: Graphing Calculator  One way to answer this question is to solve the equation for \( S \) in terms of \( x \) with \( S = 130,000 \). This is a difficult equation to solve. Although trial and error could be used, it would be easier to use the graphing calculator graph in Figure 6. Adding the line \( y = 130,000 \) to the graph and then using the capability of the calculator to find the intersection point with the curve shows the annual interest rate must be at least 8.79% to the nearest hundredth. See Figure 7 below.

Method 2: TVM Solver  Using the TVM Solver under the FINANCE menu on the TI-83/84 Plus calculator, enter 240 for \( N \) (the number of periods), 0 for \( PV \) (present value), −200 for \( PMT \) (negative because the money is being paid out), 130000 for \( FV \) (future value), and 12 for \( P/Y \) (payments per year). Put the cursor next to \( I\% \) (payment) and press SOLVE. The result, shown in Figure 8, indicates that an interest rate of 8.79% is needed.
5.2 Future Value of an Annuity

**Example 6**  
**Sinking Fund**

Suppose Sarah, in Example 5, cannot get the higher interest rate to produce $130,000 in 20 years. To meet that goal, she must increase her monthly payment. What payment should she make each month?

**Solution**  
Start with the annuity formula

\[ S = R \left( \frac{(1 + i)^n - 1}{i} \right). \]

Solve for \( R \) by multiplying both sides by \( i/(1 + i)^n - 1 \).

\[ R = \frac{Si}{(1 + i)^n - 1} \]

Now substitute \( S = 130,000 \), \( i = 0.072/12 \), and \( n = 12(20) \) to find \( R \).

\[ R = \frac{(130,000)(0.072/12)}{(1 + (0.072/12))^{12(20)} - 1} = 243.5540887 \]

Sarah will need payments of $243.56 each month for 20 years to accumulate at least $130,000. Notice that $243.55 is not quite enough, so round up here. Figure 9 shows the point of intersection of the graphs of

\[ Y_1 = X \left( \frac{(1 + 0.072/12)^{12(20)} - 1}{0.072/12} \right) \]

and \( Y_2 = 130,000 \). The result agrees with the answer we found above analytically. The table shown in Figure 9 confirms that the payment should be between $243 and $244.

We can also use a graphing calculator or spreadsheet to make a table of the amount in a sinking fund. In the formula for future value of an annuity, simply let \( n \) be a variable with values from 1 to the total number of payments. Figure 10(a)

<table>
<thead>
<tr>
<th>( n )</th>
<th>Amount in Fund</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>243.55</td>
</tr>
<tr>
<td>2</td>
<td>488.56</td>
</tr>
<tr>
<td>3</td>
<td>733.04</td>
</tr>
<tr>
<td>4</td>
<td>983.00</td>
</tr>
<tr>
<td>5</td>
<td>1232.45</td>
</tr>
<tr>
<td>6</td>
<td>1483.40</td>
</tr>
<tr>
<td>7</td>
<td>1735.85</td>
</tr>
<tr>
<td>8</td>
<td>1999.81</td>
</tr>
<tr>
<td>9</td>
<td>2264.30</td>
</tr>
<tr>
<td>10</td>
<td>2530.32</td>
</tr>
<tr>
<td>11</td>
<td>2760.89</td>
</tr>
<tr>
<td>12</td>
<td>3021.00</td>
</tr>
</tbody>
</table>
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shows the beginning of such a table generated on a TI-83/84 Plus for Example 6. Figure 10(b) shows the beginning of the same table using Microsoft Excel.

Annuities Due  The formula developed above is for ordinary annuities—those with payments made at the end of each time period. These results can be modified slightly to apply to annuities due—annuities in which payments are made at the beginning of each time period. To find the future value of an annuity due, treat each payment as if it were made at the end of the preceding period. That is, find $s_{n+i}$ for one additional period; to compensate for this, subtract the amount of one payment.

Thus, the future value of an annuity due of $n$ payments of $R$ dollars each at the beginning of consecutive interest periods, with interest compounded at the rate of $i$ per period, is

$$S = R \left( \frac{(1 + i)^{n+1} - 1}{i} \right) - R \quad \text{or} \quad S = Rs_{n+i} - R.$$

The finance feature of the TI-83/84 Plus can be used to find the future value of an annuity due as well as an ordinary annuity. If this feature is not built in, you may wish to program your calculator to evaluate this formula, too.

**EXAMPLE 7** Future Value of an Annuity Due

Find the future value of an annuity due if payments of $500 are made at the beginning of each quarter for 7 years, in an account paying 6% compounded quarterly.

**Solution** In 7 years, there are $n = 28$ quarterly periods. Add one period to get $n + 1 = 29$, and use the formula with $i = 6\%/4 = 1.5\%$.

$$S = 500 \left( \frac{(1.015)^{29} - 1}{.015} \right) - 500 = 17,499.35$$

The account will have a total of $17,499.35 after 7 years.

**5.2 EXERCISES**

Find the fifth term of each geometric sequence.

1. $a = 3; \ r = 2$  
2. $a = 5; \ r = 3$  
3. $a = -8; \ r = 3$  
4. $a = -6; \ r = 2$

5. $a = 1; \ r = -3$  
6. $a = 12; \ r = -2$  
7. $a = 1024; \ r = \frac{1}{2}$  
8. $a = 729; \ r = \frac{1}{3}$

Find the sum of the first four terms for each geometric sequence.

9. $a = 1; \ r = 2$  
10. $a = 3; \ r = 3$  
11. $a = 5; \ r = \frac{1}{5}$

12. $a = 6; \ r = \frac{1}{2}$  
13. $a = 128; \ r = \frac{3}{2}$  
14. $a = 81; \ r = -\frac{2}{3}$

---

Find each value.

15. $s_{105}$
16. $s_{106}$
17. $s_{043}$
18. $s_{015}$

19. List some reasons for establishing a sinking fund.

20. Explain the difference between an ordinary annuity and an annuity due.

Find the future value of each ordinary annuity. Interest is compounded annually.

21. $R = 100; \ i = 0.06; \ n = 4$
22. $R = 1000; \ i = 0.06; \ n = 5$
23. $R = 46,000; \ i = 0.063; \ n = 32$
24. $R = 29,500; \ i = 0.058; \ n = 15$

Find the future value of each ordinary annuity, if payments are made and interest is compounded as given.

25. $R = 9200; \ 10\% \ interest \ compounded \ semiannually \ for \ 7 \ years$
26. $R = 3700; \ 8\% \ interest \ compounded \ semiannually \ for \ 11 \ years$
27. $R = 800; \ 6.51\% \ interest \ compounded \ semiannually \ for \ 12 \ years$
28. $R = 4600; \ 8.73\% \ interest \ compounded \ quarterly \ for \ 9 \ years$
29. $R = 15,000; \ 12.1\% \ interest \ compounded \ quarterly \ for \ 6 \ years$
30. $R = 42,000; \ 10.05\% \ interest \ compounded \ semiannually \ for \ 12 \ years$

Find the future value of each annuity due. Assume that interest is compounded annually.

31. $R = 600; \ i = 0.06; \ n = 8$
32. $R = 1400; \ i = 0.08; \ n = 10$
33. $R = 20,000; \ i = 0.08; \ n = 6$
34. $R = 4000; \ i = 0.06; \ n = 11$

Find the future value of each annuity due.

35. Payments of $1000 made at the beginning of each semiannual period for 9 years at 8.15% compounded semiannually
36. $750 \ deposited \ at \ the \ beginning \ of \ each \ month \ for \ 15 \ years \ at \ 5.9\% \ compounded \ monthly$
37. $100 \ deposited \ at \ the \ beginning \ of \ each \ quarter \ for \ 9 \ years \ at \ 12.4\% \ compounded \ quarterly$
38. $1500 \ deposited \ at \ the \ beginning \ of \ each \ semiannual \ period \ for \ 11 \ years \ at \ 5.6\% \ compounded \ semiannually$

Find the periodic payment that will amount to each given sum under the given conditions.

39. $S = 10,000; \ interest \ is \ 5\% \ compounded \ annually; \ payments \ are \ made \ at \ the \ end \ of \ each \ year \ for \ 12 \ years$
40. $S = 100,000; \ interest \ is \ 8\% \ compounded \ semiannually; \ payments \ are \ made \ at \ the \ end \ of \ each \ semiannual \ period \ for \ 9 \ years$

41. What is meant by a sinking fund? Give an example of a sinking fund.

Find the amount of each payment to be made into a sinking fund so that enough will be present to accumulate the following amounts. Payments are made at the end of each period.

42. $8500; \ money \ earns \ 8\% \ compounded \ annually; \ 7 \ annual \ payments$
43. $2000; \ money \ earns \ 6\% \ compounded \ annually; \ 5 \ annual \ payments$
44. $75,000; \ money \ earns \ 6\% \ compounded \ semiannually \ for \ 4\frac{1}{2} \ years$
45. $25,000; \ money \ earns \ 5.7\% \ compounded \ quarterly \ for \ 3\frac{1}{2} \ years$
46. $50,000; \ money \ earns \ 7.9\% \ compounded \ quarterly \ for \ 2\frac{1}{2} \ years$
47. $9000; \ money \ earns \ 12.23\% \ compounded \ monthly \ for \ 2\frac{1}{2} \ years
Applications

BUSINESS AND ECONOMICS

48. **Comparing Accounts** Alex Levering deposits $12,000 at the end of each year for 9 years in an account paying 8% interest compounded annually.
   a. Find the final amount she will have on deposit.
   b. Alex’s brother-in-law works in a bank that pays 6% compounded annually. If she deposits money in this bank instead of the one above, how much will she have in her account?
   c. How much would Alex lose over 9 years by using her brother-in-law’s bank?

49. **Savings** Tom De Marco is saving for a computer. At the end of each month he puts $60 in a savings account that pays 8% interest compounded monthly. How much is in the account after 3 years?

50. **Savings** Hassi is paid on the first day of the month and $80 is automatically deducted from his pay and deposited in a savings account. If the account pays 7.5% interest compounded monthly, how much will be in the account after 3 years and 9 months?

51. **Savings** A typical pack-a-day smoker spends about $55 per month on cigarettes. Suppose the smoker invests that amount each month in a savings account at 4.8% interest compounded monthly. What would the account be worth after 40 years?

52. **Savings** A father opened a savings account for his daughter on the day she was born, depositing $1000. Each year on her birthday he deposits another $1000, making the last deposit on her twenty-first birthday. If the account pays 9.5% interest compounded annually, how much will be in the account after 3 years and 9 months?

53. **Retirement Planning** A 45-year-old man puts $1000 in a retirement account at the end of each quarter until he reaches the age of 60 and makes no further deposits. If the account pays 8% interest compounded quarterly, how much will be in the account when the man retires at age 65?

54. **Retirement Planning** At the end of each quarter a 50-year-old woman puts $1200 in a retirement account that pays 7% interest compounded quarterly. When she reaches age 60, she withdraws the entire amount and places it in a mutual fund that pays 9% interest compounded monthly. From then on she deposits $300 in the mutual fund at the end of each month. How much is in the account when she reaches age 65?

55. **Savings** Jasspreet Kaur deposits $2435 at the beginning of each semiannual period for 8 years in an account paying 6% compounded semiannually. She then leaves that money alone, with no further deposits, for an additional 5 years. Find the final amount on deposit after the entire 13-year period.

56. **Savings** Chuck Hickman deposits $10,000 at the beginning of each year for 12 years in an account paying 5% compounded annually. He then puts the total amount on deposit in another account paying 6% compounded semiannually for another 9 years. Find the final amount on deposit after the entire 21-year period.

57. **Savings** Greg Tobin needs $10,000 in 8 years.
   a. What amount should he deposit at the end of each quarter at 8% compounded quarterly so that he will have his $10,000?
   b. Find Greg’s quarterly deposit if the money is deposited at 6% compounded quarterly.

58. **Buying Equipment** Harv, the owner of Harv’s Meats, knows that he must buy a new deboner machine in 4 years. The machine costs $12,000. In order to accumulate enough money to pay for the machine, Harv decides to deposit a sum of money at the end of each 6 months in an account paying 6% compounded semiannually. How much should each payment be?

59. **Buying a Car** Susan Laferriere wants to buy an $18,000 car in 6 years. How much money must she deposit at the end of each quarter in an account paying 5% compounded quarterly so that she will have enough to pay for her car?

---

**Individual Retirement Accounts** Suppose a 40-year-old person deposits $2000 per year in an Individual Retirement Account until age 65. Find the total in the account with the following assumptions of interest rates. (Assume semiannual compounding, with payments of $1000 made at the end of each semiannual period.)

60. 6%
61. 8%
62. 4%
63. 10%
5.3 Present Value of an Annuity; Amortization

In Exercises 64 and 65, use a graphing calculator to find the value of \( t \) that produces the given value of \( S \). (See Example 5(b).)

64. Retirement To save for retirement, Karla Harby put $300 each month into an ordinary annuity for 20 years. Interest was compounded monthly. At the end of the 20 years, the annuity was worth $147,126. What annual interest rate did she receive?

65. Rate of Return Jennifer Wall made payments of $250 per month at the end of each month to purchase a piece of property. At the end of 30 years, she completely owned the property, which she sold for $330,000. What annual interest rate would she need to earn on an annuity for a comparable rate of return?

66. Lottery In a 1992 Virginia lottery, the jackpot was $27 million. An Australian investment firm tried to buy all possible combinations of numbers, which would have cost $7 million. In fact, the firm ran out of time and was unable to buy all combinations, but ended up with the only winning ticket anyway. The firm received the jackpot in 20 equal annual payments of $1.35 million.* Assume these payments meet the conditions of an ordinary annuity.

a. Suppose the firm can invest money at 8% interest compounded annually. How many years would it take until the investors would be further ahead than if they had simply invested the $7 million at the same rate? (Hint: Experiment with different values of \( n \), the number of years, or use a graphing calculator to plot the value of both investments as a function of the number of years.)

b. How many years would it take in part a at an interest rate of 12%?

67. Buying Real Estate Marisa Raffaele sells some land in Nevada. She will be paid a lump sum of $60,000 in 7 years. Until then, the buyer pays 8% simple interest quarterly.

a. Find the amount of each quarterly interest payment on the $60,000.

b. The buyer sets up a sinking fund so that enough money will be present to pay off the $60,000. The buyer will make semiannual payments into the sinking fund; the account pays 6% compounded semiannually. Find the amount of each payment into the fund.

68. Buying Rare Stamps Paul Altier bought a rare stamp for his collection. He agreed to pay a lump sum of $4000 after 5 years. Until then, he pays 6% simple interest semiannually on the $4000.

a. Find the amount of each semiannual interest payment.

b. Paul sets up a sinking fund so that enough money will be present to pay off the $4000. He will make annual payments into the fund. The account pays 8% compounded annually. Find the amount of each payment.

69. Down Payment A conventional loan, such as for a car or a house, is similar to an annuity, but usually includes a down payment. Show that if a down payment of \( D \) dollars is made at the beginning of the loan period, the future value of all the payments, including the down payment, is

\[
S = D(1 + i)^n + R \left[ \frac{(1 + i)^n - 1}{i} \right].
\]

5.3 PRESENT VALUE OF AN ANNUITY; AMORTIZATION

**THINK ABOUT IT**

What monthly payment will pay off a $10,000 car loan in 36 monthly payments at 12% annual interest?

The answer to this question is given later in this section. We shall see that it involves finding the present value of an annuity.

Suppose that at the end of each year, for the next 10 years, $500 is deposited in a savings account paying 7% interest compounded annually. This is an example of an ordinary annuity. The present value of an annuity is the amount that would have to be deposited in one lump sum today (at the same compound interest rate) in order to produce exactly the same balance at the end of 10 years. We can find a formula for the present value of an annuity as follows.

Suppose deposits of \( R \) dollars are made at the end of each period for \( n \) periods at interest rate \( i \) per period. Then the amount in the account after \( n \) periods is the future value of this annuity:

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I

On the other hand, if \( P \) dollars are deposited today at the same compound interest rate \( i \), then at the end of \( n \) periods, the amount in the account is \( P(1 + i)^n \). If \( P \) is the present value of the annuity, this amount must be the same as the amount \( S \) in the formula above; that is,

\[
P(1 + i)^n = R \left[ \frac{(1 + i)^n - 1}{i} \right].
\]

To solve this equation for \( P \), multiply both sides by \( (1 + i)^{-n} \).

\[
P = R(1 + i)^{-n} \left[ \frac{(1 + i)^n - 1}{i} \right]
\]

Use the distributive property; also recall that \( (1 + i)^{-n}(1 + i)^n = 1 \).

\[
\frac{S}{\left[ \frac{(1 + i)^n - 1}{i} \right]} = R \frac{1}{i}
\]

The amount \( P \) is the present value of the annuity. The quantity in brackets is abbreviated as \( a_{\overline{n}|i} \), so

\[
a_{\overline{n}|i} = \frac{1 - (1 + i)^{-n}}{i}.
\]

(The symbol \( a_{\overline{n}|i} \) is read “(a-angle-n) at \( i \).” Compare this quantity with \( S_{\overline{n}|i} \) in the previous section.) The formula for the present value of an annuity is summarized below.

**PRESENT VALUE OF AN ANNUITY**

The present value \( P \) of an annuity of \( n \) payments of \( R \) dollars each at the end of consecutive interest periods with interest compounded at a rate of interest \( i \) per period is

\[
P = R \left[ 1 - \left( \frac{1 + i}{i} \right)^{-n} \right]
\]

or

\[
P = Ra_{\overline{n}|i}
\]

**CAUTION** Don’t confuse the formula for the present value of an annuity with the one for the future value of an annuity. Notice the difference: the numerator of the fraction in the present value formula is \( 1 - (1 + i)^{-n} \), but in the future value formula, it is \( (1 + i)^n - 1 \).

The financial feature of the TI-83/84 Plus calculator can be used to find the present value of an annuity by choosing that option from the menu and entering the required information. If your calculator does not have this built-in feature, it will be useful to store a program to calculate present value of an annuity in your calculator. A program is given in *The Graphing Calculator Manual* that is available with this book.

**EXAMPLE 1**  Present Value of an Annuity

Mr. Bryer and Ms. Gonzalez are both graduates of the Brisbane Institute of Technology. They both agree to contribute to the endowment fund of BIT. Mr. Bryer
sends that he will give $500 at the end of each year for 9 years. Ms. Gonsalez prefers to give a lump sum today. What lump sum can she give that will equal the present value of Mr. Bryer’s annual gifts, if the endowment fund earns 7.5% compounded annually?

**Solution** Here, \( R = 500, n = 9, \) and \( i = .075, \) and we have

\[
P = R \cdot a_{i0.75} = 500 \left[ \frac{1 - (1.075)^{-9}}{.075} \right] \approx 3189.44.
\]

Therefore, Ms. Gonsalez must donate a lump sum of $3189.44 today.

One of the most important uses of annuities is in determining the equal monthly payments needed to pay off a loan, as illustrated in the next example.

**Example 2**  
**Car Payments**  
A car costs $12,000. After a down payment of $2000, the balance will be paid off in 36 equal monthly payments with interest of 12% per year on the unpaid balance. Find the amount of each payment.

**Solution** A single lump sum payment of $10,000 today would pay off the loan. So, $10,000 is the present value of an annuity of 36 monthly payments with interest of 12%/12 = 1% per month. Thus, \( P = 10,000, n = 36, i = .01, \) and we must find the monthly payment \( R \) in the formula

\[
P = \frac{1 - (1 + i)^{-n}}{i}
\]

\[
10,000 = \frac{1 - (1.01)^{-36}}{.01}
\]

\[
R \approx 332.1430981.
\]

A monthly payment of $332.14 will be needed.

Each payment in Example 2 includes interest on the unpaid balance with the remainder going to reduce the loan. For example, the first payment of $332.14 includes interest of \( .01(10,000) = 100 \) and is divided as follows.

<table>
<thead>
<tr>
<th>monthly payment due to reduce balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>$332.15 - $100 = $232.15</td>
</tr>
</tbody>
</table>

At the end of this section, amortization schedules show that this procedure does reduce the loan to $0 after all payments are made (the final payment may be slightly different).

**Amortization** A loan is amortized if both the principal and interest are paid by a sequence of equal periodic payments. In Example 2, a loan of $10,000 at 12% interest compounded monthly could be amortized by paying $332.14 per month for 36 months.

The periodic payment needed to amortize a loan may be found, as in Example 2, by solving the present value equation for \( R \).
Chapter 5  Mathematics of Finance

**AMORTIZATION PAYMENTS**

A loan of \( P \) dollars at interest rate \( i \) per period may be amortized in \( n \) equal periodic payments of \( R \) dollars made at the end of each period, where

\[
R = \frac{P}{a_{\frac{i}{12n}}} = \frac{P}{\frac{1}{i} \left( 1 + \frac{i}{12} \right)^{-n}} = \frac{Pi}{1 - (1 + i)^{-n}}.
\]

**EXAMPLE 3**  

**Home Mortgage**

The Perez family buys a house for $94,000 with a down payment of $16,000. They take out a 30-year mortgage for $78,000 at an annual interest rate of 9.6%.

(a) Find the amount of the monthly payment needed to amortize this loan.

**Solution**  Here \( P = 78,000 \) and the monthly interest rate is \( 9.6\%/12 = .096/12 = .008.* \) The number of monthly payments is \( 12 \cdot 30 = 360. \) Therefore,

\[
R = \frac{78,000}{a_{0.008}} = \frac{78,000}{1 - (1.008)^{-360}} \approx 661.56.
\]

Monthly payments of $661.56 are required to amortize the loan.

(b) Find the total amount of interest paid when the loan is amortized over 30 years.

**Solution**  The Perez family makes 360 payments of $661.56 each, for a total of $238,161.60. Since the amount of the loan was $78,000, the total interest paid is

\[
$238,161.60 - $78,000 = $160,161.60.
\]

This large amount of interest is typical of what happens with a long mortgage. A 15-year mortgage would have higher payments, but would involve significantly less interest.

(c) Find the part of the first payment that is interest and the part that is applied to reducing the debt.

**Solution**  During the first month, the entire $78,000 is owed. Interest on this amount for 1 month is found by the formula for simple interest, with \( r = \) annual interest rate and \( t = \) time in years.

\[
I = Prt = 78,000(0.096)\frac{1}{12} = 624
\]

At the end of the month, a payment of $661.56 is made; since $624 of this is interest, a total of

\[
$661.56 - $624 = $37.56
\]

is applied to the reduction of the original debt.

---

*Mortgage rates are quoted in terms of annual interest, but it is always understood that the monthly rate is 1/12 of the annual rate and that interest is compounded monthly.
5.3 Present Value of an Annuity; Amortization

It can be shown that the unpaid balance after \( x \) payments is approximately given by the function

\[
y = R \left[ \frac{1 - (1 + i)^{-(n-x)}}{i} \right].
\]

For example, the unpaid balance in Example 3 after 1 payment is

\[
y = \frac{80,000 \left[ 1 - (1.008)^{-359} \right]}{.008} = \$77,961.87.
\]

This is very close to the amount left after deducting the $37.56 applied to the loan in part (c):

\[
\$78,000 - \$37.56 = \$77,962.44.
\]

A calculator graph of this function is shown in Figure 11.

We can find the unpaid balance after any number of payments, \( x \), by finding the \( y \)-value that corresponds to \( x \). For example, the remaining balance after 5 years or 60 payments is shown at the bottom of the window in Figure 12(a). You may be surprised that the remaining balance on a $78,000 loan is as large as $75,121.10. This is because most of the early payments on a loan go toward interest, as we saw in Example 3(c).

By adding the graph of \( y = (1/2)78,000 = 39,000 \) to the figure, we can find when half the loan has been repaid. From Figure 12(b) we see that 280 payments are required. Note that only 80 payments remain at that point, which again emphasizes the fact that the earlier payments do little to reduce the loan.

**Amortization Schedules**  In the preceding example, 360 payments are made to amortize a $78,000 loan. The loan balance after the first payment is reduced by only $37.56, which is much less than \((1/360)(78,000) = \$216.67\). Therefore, even though equal payments are made to amortize a loan, the loan balance does not decrease in equal steps. This fact is very important if a loan is paid off early.

**Example 4 Early Payment**

Susan Dratch borrows $1000 for 1 year at 12% annual interest compounded monthly. Verify that her monthly loan payment is $88.85. After making three payments, she decides to pay off the remaining balance all at once. How much must she pay?
Solution Since nine payments remain to be paid, they can be thought of as an annuity consisting of nine payments of $88.85 at 1% interest per period. The present value of this annuity is

$$88.85 \left[ \frac{1 - (1.01)^{-9}}{.01} \right] \approx 761.09.$$  

So Susan’s remaining balance, computed by this method, is $761.09.

An alternative method of figuring the balance is to consider the payments already made as an annuity of three payments. At the beginning, the present value of this annuity was

$$88.85 \left[ \frac{1 - (1.01)^{-3}}{.01} \right] \approx 261.31.$$  

So she still owes the difference $1000 - 261.31 = $738.69. Furthermore, she owes the interest on this amount for 3 months, for a total of

$$(738.69)(1.01)^3 = $761.07.$$  

This balance due differs from the one obtained by the first method by 2 cents because the monthly payment and the other calculations were rounded to the nearest penny.

Although most people would not quibble about a difference of 2 cents in the balance due in Example 4, the difference in other cases (larger amounts or longer terms) might be more than that. A bank or business must keep its books accurately to the nearest penny, so it must determine the balance due in such cases unambiguously and exactly. This is done by means of an amortization schedule, which lists how much of each payment is interest and how much goes to reduce the balance, as well as how much is owed after each payment.

**Example 5**

**Amortization Table**

Determine the exact amount Susan Dratch in Example 4 owes after three monthly payments.

**Solution** An amortization table for the loan is shown on the next page. It is obtained as follows. The annual interest rate is 12% compounded monthly, so the interest rate per month is $12%/12 = 1% = .01$. When the first payment is made, 1 month’s interest—namely $.01(1000) = $10—is owed. Subtracting this from the $88.85 payment leaves $78.85 to be applied to repayment. Hence, the principal at the end of the first payment period is $1000 - 78.85 = $921.15, as shown in the “payment 1” line of the chart.

When payment 2 is made, 1 month’s interest on $921.15 is owed, namely $.01(921.15) = $9.21. Subtracting this from the $88.85 payment leaves $79.64 to reduce the principal. Hence, the principal at the end of payment 2 is $921.15 - 79.64 = $841.51. The interest portion of payment 3 is based on this amount, and the remaining lines of the table are found in a similar fashion.

The schedule shows that after three payments, she still owes $761.08, an amount that differs slightly from that obtained by either method in Example 4.
Payment Number | Amount of Payment | Interest for Period | Portion to Principal | Principal at End of Period
---|---|---|---|---
0 | — | — | — | $1000.00
1 | $88.85 | $10.00 | $78.85 | $921.15
2 | $88.85 | $9.21 | $79.64 | $841.51
3 | $88.85 | $8.42 | $80.43 | $761.08
4 | $88.85 | $7.61 | $81.24 | $679.84
5 | $88.85 | $6.80 | $82.05 | $597.79
6 | $88.85 | $5.98 | $82.87 | $514.92
7 | $88.85 | $5.15 | $83.70 | $431.22
8 | $88.85 | $4.31 | $84.54 | $346.68
9 | $88.85 | $3.47 | $85.38 | $261.30
10 | $88.85 | $2.61 | $86.24 | $175.06
11 | $88.85 | $1.75 | $87.10 | $87.96
12 | $88.84 | $.88 | $87.96 | 0

The amortization schedule in Example 5 is typical. In particular, note that all payments are the same except the last one. It is often necessary to adjust the amount of the final payment to account for rounding off earlier, and to ensure that the final balance is exactly 0.

An amortization schedule also shows how the periodic payments are applied to interest and principal. The amount going to interest decreases with each payment, while the amount going to reduce the principal increases with each payment.

A graphing calculator program to produce an amortization schedule is available in The Graphing Calculator Manual that is available with this book. The TI-83/84 Plus includes a built-in program to find the amortization payment. Spreadsheets are another useful tool for creating amortization tables. Microsoft Excel has a built-in feature for calculating monthly payments. Figure 13 shows an Excel amortization table for Example 5. For more details, see The Spreadsheet Manual, also available with this book.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
</tr>
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<tr>
<td>0</td>
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<td>—</td>
<td>—</td>
<td>—</td>
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<td>2</td>
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<tr>
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<td>83.70</td>
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<tr>
<td>8</td>
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<td>4.31</td>
<td>84.54</td>
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<tr>
<td>9</td>
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<td>3.47</td>
<td>85.38</td>
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<tr>
<td>10</td>
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<td>175.05</td>
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</tr>
<tr>
<td>11</td>
<td>88.85</td>
<td>1.75</td>
<td>87.10</td>
<td>87.96</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>88.84</td>
<td>0.88</td>
<td>87.97</td>
<td>-0.02</td>
<td></td>
</tr>
</tbody>
</table>

**FIGURE 13**
5.3 Exercises

1. Which of the following is represented by \( a_{n|}\)?
   \[
   \begin{align*}
   \text{a. } & \frac{(1 + i)^n - 1}{i} \\
   \text{b. } & \frac{(1 + i)^n - 1}{i} \\
   \text{c. } & \frac{1 - (1 + i)^{-n}}{i} \\
   \text{d. } & \frac{1 - (1 + i)^n}{i}
   \end{align*}
   \]

2. Which of the choices in Exercise 1 represents \( s_{n|}\)?

Find each value.
3. \( a_{10.026} \)  
4. \( a_{10.033} \)  
5. \( a_{10.045} \)  
6. \( a_{10.029} \)

7. Explain the difference between the present value of an annuity and the future value of an annuity. For a given annuity, which is larger? Why?

Find the present value of each ordinary annuity.
8. Payments of $890 each year for 16 years at 8% compounded annually
9. Payments of $1400 each year for 8 years at 8% compounded annually
10. Payments of $10,000 semiannually for 15 years at 10% compounded semiannually
11. Payments of $50,000 quarterly for 10 years at 8% compounded quarterly
12. Payments of $15,806 quarterly for 3 years at 10.8% compounded quarterly
13. Payments of $18,579 every 6 months for 8 years at 9.4% compounded semiannually

Find the lump sum deposited today that will yield the same total amount as payments of $10,000 at the end of each year for 15 years at each of the given interest rates.
14. 4% compounded annually
15. 6% compounded annually

16. What does it mean to amortize a loan?

Find the payment necessary to amortize each loan.
17. $2500; 8% compounded quarterly; 6 quarterly payments
18. $41,000; 10% compounded semiannually; 10 semiannual payments
19. $90,000; 8% compounded annually; 12 annual payments
20. $140,000; 12% compounded quarterly; 15 quarterly payments
21. $7400; 8.2% compounded semiannually; 18 semiannual payments
22. $5500; 12.5% compounded monthly; 24 monthly payments

Use the amortization table in Example 5 to answer the questions in Exercises 23–26.

23. How much of the fourth payment is interest?
24. How much of the eleventh payment is used to reduce the debt?
25. How much interest is paid in the first 4 months of the loan?
26. How much interest is paid in the last 4 months of the loan?

27. What sum deposited today at 5% compounded annually for 8 years will provide the same amount as $1000 deposited at the end of each year for 8 years at 6% compounded annually?
28. What lump sum deposited today at 8% compounded quarterly for 10 years will yield the same final amount as deposits of $4000 at the end of each 6-month period for 10 years at 6% compounded semiannually?

Find the monthly house payments necessary to amortize each loan.

29. $149,560 at 7.75% for 25 years
30. $170,892 at 8.11% for 30 years
31. $153,762 at 8.45% for 30 years
32. $96,511 at 9.57% for 25 years

Applications

BUSINESS AND ECONOMICS

33. **House Payments** Calculate the monthly payment and total amount of interest paid in Example 3 with a 15-year loan, and then compare with the results of Example 3.

34. **Installment Buying** Stereo Shack sells a stereo system for $600 down and monthly payments of $30 for the next 3 years. If the interest rate is 1.25% per month on the unpaid balance, find:
   a. the cost of the stereo system;
   b. the total amount of interest paid.

35. **Car Payments** Hong Le buys a car costing $6000. He agrees to make payments at the end of each monthly period for 4 years. He pays 12% interest, compounded monthly.
   a. What is the amount of each payment?
   b. Find the total amount of interest Le will pay.

36. **Land Purchase** A speculator agrees to pay $15,000 for a parcel of land; this amount, with interest, will be paid over 4 years, with semiannual payments, at an interest rate of 10% compounded semiannually. Find the amount of each payment.

37. **New Car** General Motors’ “Summerdrive 2 the Max” advertising campaign pledged a cash-back allowance of $3500 or 0% financing for 60 months for a 2003 Pontiac Sunbird car.
   a. Determine the payments on a Sunbird if a person chooses the 0% financing option and needs to finance $13,500 for 60 months.
   b. Determine the payments on a Sunbird if a person chooses the cash-back option and now needs to finance only $10,000. Assume that the buyer is able to find financing from a local bank at 5.9% for 60 months, compounded monthly.
   c. Discuss which deal is best and why.
   d. Find the interest rate at the bank that would make the other option optimal.

38. **New Truck** General Motors’ “Summerdrive 2 the Max” advertising campaign pledged a cash-back allowance of $4000 or 0% financing for 60 months for a 2003 Chevrolet Avalanche pickup truck.
   a. Determine the payments on an Avalanche if a person chooses the 0% financing option and needs to finance $25,000 for 60 months.
   b. If a person purchases an Avalanche and chooses the cash-back option, she will need to finance $21,500. Assume that she is able to choose between two options at her local bank, 4.5% for 48 months or 6.9% for 60 months. Find the monthly payment and the total amount of money that she will pay back to the bank on each option.
   c. Of the three deals, discuss which is best and why.

39. **Lottery Winnings** In most states, the winnings of million-dollar lottery jackpots are divided into equal payments given annually for 20 years. (In Colorado, the results are distributed over 25 years.) This means that the present value of the jackpot is worth less than the stated prize, with the actual value determined by the interest rate at which the money could be invested.

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†Ibid.
a. Find the present value of a $1 million lottery jackpot distributed in equal annual payments over 20 years, using an interest rate of 5%.

b. Find the present value of a $1 million lottery jackpot distributed in equal annual payments over 20 years, using an interest rate of 9%.

c. Calculate the answer for part a using the 25-year distribution time in Colorado.

d. Calculate the answer for part b using the 25-year distribution time in Colorado.

46. Loan Payments When Nancy Hart opened her law office, she bought $14,000 worth of law books and $7200 worth of office furniture. She paid $1200 down and agreed to amortize the balance with semiannual payments for 5 years, at 12% compounded semiannually.

a. Find the amount of each payment.

b. Refer to the text and Figures 11 and 12. When her loan had been reduced below $5000, Nancy received a large tax refund and decided to pay off the loan. How many payments were left at this time?

47. House Payments Kareem Adiagbo buys a house for $285,000. He pays $60,000 down and takes out a mortgage at 9.5% on the balance. Find his monthly payment and the total amount of interest he will pay if the length of the mortgage is

a. 15 years;

b. 20 years;

c. 25 years.

d. Refer to the text and Figures 11 and 12. When will half the 20-year loan in part b be paid off?

48. Inheritance Sandi Goldstein has inherited $25,000 from her grandfather’s estate. She deposits the money in an account offering 6% interest compounded annually. She wants to make equal annual withdrawals from the account so that the money (principal and interest) lasts exactly 8 years.

a. Find the amount of each withdrawal.

b. Find the amount of each withdrawal if the money must last 12 years.

49. Charitable Trust The trustees of a college have accepted a gift of $150,000. The donor has directed the trustees to

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deposit the money in an account paying 6% per year, compounded semiannually. The trustees may make equal withdrawals at the end of each 6-month period; the money must last 5 years.

a. Find the amount of each withdrawal.
b. Find the amount of each withdrawal if the money must last 6 years.

**Amortization** Prepare an amortization schedule for each loan.

50. A loan of $37,947.50 with interest at 8.5% compounded annually, to be paid with equal annual payments over 10 years.

51. A loan of $4835.80 at 9.25% interest compounded semi-annually, to be repaid in 5 years in equal semiannual payments.

**Perpetuity** A perpetuity is an annuity in which the payments go on forever. We can derive a formula for the present value of a perpetuity by taking the formula for the present value of an annuity and looking at what happens when \( n \) gets larger and larger. Explain why the present value of an annuity is given by

\[
P = \frac{R}{i}.
\]

53. **Perpetuity** Using the result of Exercise 52, find the present value of perpetuities for each of the following.

a. Payments of $1000 a year with 4% interest compounded annually
b. Payments of $600 every 3 months with 6% interest compounded quarterly

**CHAPTER SUMMARY**

This chapter introduces the mathematics of finance. Simple interest is the starting point; when interest is earned on interest previously earned, we have compound interest. In an annuity, money continues to be deposited at regular intervals, and compound interest is earned on that money. In an ordinary annuity, the compounding period is the same as the time between payments, which simplifies the calculations. An annuity due is slightly different, in that the payments are made at the beginning of each time period. A sinking fund is like an annuity; a fund is set up to receive periodic payments, so the payments plus the compound interest will produce a desired sum by a certain date. The present value of an annuity is the amount that would have to be deposited today to produce the same amount as the annuity at the end of a specified time. This idea leads to an amortization table for a loan, which shows the payments, broken down into interest and principal, for a loan to be paid back after a specified time.

**A Strategy for Solving Finance Problems**

We have presented a lot of new formulas in this chapter. By answering the following questions, you can decide which formula to use for a particular problem.

1. Is simple or compound interest involved?
   
   Simple interest is normally used for investments or loans of a year or less; compound interest is normally used in all other cases.

2. If simple interest is being used, what is being sought: interest amount, future value, present value, or interest rate?

3. If compound interest is being used, does it involve a lump sum (single payment) or an annuity (sequence of payments)?
   
   a. For a lump sum, what is being sought: present value, future value, number of periods at interest, or effective rate?
   
   b. For an annuity,
      
      i. Is it an ordinary annuity (payment at the end of each period) or an annuity due (payment at the beginning of each period)?
      
      ii. What is being sought: present value, future value, or payment amount?
Once you have answered these questions, choose the appropriate formula and work the problem. As a final step, consider whether the answer you get makes sense. For instance, present value should always be less than future value. The amount of interest or the payments in an annuity should be fairly small compared to the total future value.

**List of Variables**

- $r$ is the annual interest rate.
- $i$ is the interest rate per period.
- $t$ is the number of years.
- $n$ is the number of periods.
- $m$ is the number of periods per year.
- $P$ is the principal or present value.
- $A$ is the future value of a lump sum.
- $S$ is the future value of an annuity.
- $R$ is the periodic payment in an annuity.

\[
\frac{r}{m} \quad n = tm
\]

<table>
<thead>
<tr>
<th>Simple Interest</th>
<th>Compound Interest</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Interest</strong></td>
<td>$I = Prt$</td>
</tr>
<tr>
<td><strong>Future Value</strong></td>
<td>$A = P(1 + rt)$</td>
</tr>
<tr>
<td><strong>Present Value</strong></td>
<td>$P = \frac{A}{1 + rt}$</td>
</tr>
</tbody>
</table>

**Effective Rate**

\[
r_e = \left(1 + \frac{r}{m}\right)^m - 1
\]

**Ordinary Annuity**

- Future Value: $S = R \left[ \frac{(1 + i)^n - 1}{i} \right] = R \cdot s_{\bar{n}|i}$
- Present Value: $P = R \left[ \frac{1 - (1 + i)^{-n}}{i} \right] = R \cdot a_{\bar{n}|i}$

**Annuity Due**

- Future Value: $S = R \left[ \frac{(1 + i)^{n+1} - 1}{i} \right] - R$

**KEY TERMS**

- 5.1 simple interest
- principal
- rate
- time
- future value
- maturity value
- compound interest
- compound amount
- nominal (stated) rate
- effective rate
- present value
- geometric sequence
- terms
- common ratio
- annuity
- ordinary annuity
- future value of an ordinary annuity
- amortize a loan
- amortization schedule
- sinking fund
- annuity due
- future value of an annuity due
- present value of an annuity
CHAPTER 5 REVIEW EXERCISES

Find the simple interest for each loan.

1. $15,903 at 8% for 8 months
2. $4902 at 9.5% for 11 months
3. $42,368 at 5.22% for 5 months
4. $3478 at 7.4% for 88 days (assume a 360-day year)

5. For a given amount of money at a given interest rate for a given time period, does simple interest or compound interest produce more interest?

Find the compound amount in each loan.

6. $2800 at 6% compounded annually for 10 years
7. $19,456.11 at 12% compounded semiannually for 7 years
8. $312.45 at 6% compounded semiannually for 16 years
9. $57,809.34 at 12% compounded quarterly for 5 years

Find the amount of interest earned by each deposit.

10. $3954 at 8% compounded annually for 12 years
11. $12,699.36 at 10% compounded semiannually for 7 years
12. $12,903.45 at 10.37% compounded quarterly for 29 quarters
13. $34,677.23 at 9.72% compounded monthly for 32 months
14. What is meant by the present value of an amount A?

Find the present value of each amount.

15. $42,000 in 7 years, 12% compounded monthly
16. $17,650 in 4 years, 8% compounded quarterly
17. $1347.89 in 3.5 years, 6.77% compounded semiannually
18. $2388.90 in 44 months, 5.93% compounded monthly
19. Write the first five terms of the geometric sequence with \( a = 2 \) and \( r = 3 \).
20. Write the first four terms of the geometric sequence with \( a = 4 \) and \( r = 1/2 \).
21. Find the sixth term of the geometric sequence with \( a = -3 \) and \( r = 2 \).
22. Find the fifth term of the geometric sequence with \( a = -2 \) and \( r = -2 \).
23. Find the sum of the first four terms of the geometric sequence with \( a = -3 \) and \( r = 3 \).
24. Find the sum of the first five terms of the geometric sequence with \( a = 8000 \) and \( r = -1/2 \).
25. Find \( s_{210.04} \).
26. Find \( s_{210.05} \).
27. What is meant by the future value of an annuity?

Find the future value of each annuity.

28. $500 deposited at the end of each 6-month period for 8 years; money earns 6% compounded semiannually
29. $1288 deposited at the end of each year for 14 years; money earns 8% compounded annually
30. $4000 deposited at the end of each quarter for 7 years; money earns 6% compounded quarterly
31. $233 deposited at the end of each month for 4 years; money earns 12% compounded monthly
32. $672 deposited at the beginning of each quarter for 7 years; money earns 8% compounded quarterly
Chapter 5  Mathematics of Finance

33. $11,900 deposited at the beginning of each month for 13 months; money earns 12% compounded monthly

34. What is the purpose of a sinking fund?

Find the amount of each payment that must be made into a sinking fund to accumulate each amount. (Recall, in a sinking fund, payments are made at the end of every interest period.)

35. $6500; money earns 8% compounded annually; 6 annual payments
36. $57,000; money earns 6% compounded semiannually for $5\frac{1}{2}$ years
37. $233,188; money earns 9.7% compounded quarterly for $7\frac{1}{2}$ years
38. $1,056,788; money earns 8.12% compounded monthly for $4\frac{1}{2}$ years

Find the present value of each ordinary annuity.

39. Deposits of $850 annually for 4 years at 8% compounded annually
40. Deposits of $1500 quarterly for 7 years at 8% compounded quarterly
41. Payments of $4210 semiannually for 8 years at 8.6% compounded semiannually
42. Payments of $877.34 monthly for 17 months at 9.4% compounded monthly
43. Give two examples of the types of loans that are commonly amortized.

Find the amount of the payment necessary to amortize each loan.

44. $80,000; 8% compounded annually; 9 annual payments
45. $3200; 8% compounded quarterly; 10 quarterly payments
46. $32,000; 9.4% compounded quarterly; 17 quarterly payments
47. $51,607; 13.6% compounded monthly; 32 monthly payments

Find the monthly house payments for each mortgage.

48. $56,890 at 10.74% for 25 years
49. $77,110 at 11.45% for 30 years

A portion of an amortization table is given below for a $127,000 loan at 8.5% interest compounded monthly for 25 years.

<table>
<thead>
<tr>
<th>Payment Number</th>
<th>Amount of Payment</th>
<th>Interest for Period</th>
<th>Portion to Principal</th>
<th>Principal at End of Period</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1022.64</td>
<td>$899.58</td>
<td>$123.06</td>
<td>$126,876.94</td>
</tr>
<tr>
<td>2</td>
<td>$1022.64</td>
<td>$898.71</td>
<td>$123.93</td>
<td>$126,753.01</td>
</tr>
<tr>
<td>3</td>
<td>$1022.64</td>
<td>$897.83</td>
<td>$124.81</td>
<td>$126,628.20</td>
</tr>
<tr>
<td>4</td>
<td>$1022.64</td>
<td>$896.95</td>
<td>$125.69</td>
<td>$126,502.51</td>
</tr>
<tr>
<td>5</td>
<td>$1022.64</td>
<td>$896.06</td>
<td>$126.58</td>
<td>$126,375.93</td>
</tr>
<tr>
<td>6</td>
<td>$1022.64</td>
<td>$895.16</td>
<td>$127.48</td>
<td>$126,248.45</td>
</tr>
<tr>
<td>7</td>
<td>$1022.64</td>
<td>$894.26</td>
<td>$128.38</td>
<td>$126,120.07</td>
</tr>
<tr>
<td>8</td>
<td>$1022.64</td>
<td>$893.35</td>
<td>$129.29</td>
<td>$125,990.78</td>
</tr>
<tr>
<td>9</td>
<td>$1022.64</td>
<td>$892.43</td>
<td>$130.21</td>
<td>$125,860.57</td>
</tr>
<tr>
<td>10</td>
<td>$1022.64</td>
<td>$891.51</td>
<td>$131.13</td>
<td>$125,729.44</td>
</tr>
<tr>
<td>11</td>
<td>$1022.64</td>
<td>$890.58</td>
<td>$132.06</td>
<td>$125,597.38</td>
</tr>
<tr>
<td>12</td>
<td>$1022.64</td>
<td>$889.65</td>
<td>$132.99</td>
<td>$125,464.39</td>
</tr>
</tbody>
</table>

Use the table to answer the following questions.

50. How much of the fifth payment is interest?
51. How much of the twelfth payment is used to reduce the debt?
52. How much interest is paid in the first 3 months of the loan?
53. How much has the debt been reduced at the end of the first year?

Applications

BUSINESS AND ECONOMICS

54. Personal Finance Michael Garbin owes $5800 to his mother. He has agreed to repay the money in 10 months at an interest rate of 10.3%. How much will he owe in 10 months? How much interest will he pay?

55. Business Financing John Remington needs to borrow $9820 to buy new equipment for his business. The bank charges him 12.1% for a 7-month loan. How much interest will he be charged? What amount must he pay in 7 months?

56. Business Financing An accountant loans $28,000 at simple interest to her business. The loan is at 11.5% and earns $3255 interest. Find the time of the loan in months.

57. Business Investment A developer deposits $84,720 for 7 months and earns $4055.46 in simple interest. Find the interest rate.

58. Personal Finance In 3 years Joan McKee must pay a pledge of $7500 to her college’s building fund. What lump sum can she deposit today, at 10% compounded semiannually, so that she will have enough to pay the pledge?

59. Personal Finance Tom, a graduate student, is considering investing $500 now, when he is 23, or waiting until he is 40 to invest $500. How much more money will he have at the age of 65 if he invests now, given that he can earn 5% interest compounded quarterly?

60. Pensions Pension experts recommend that you start drawing at least 40% of your full pension as early as possible.* Suppose you have built up a pension of $12,000-annual payments by working 10 years for a company. When you leave to accept a better job, the company gives you the option of collecting half of the full pension when you reach age 55 or the full pension at age 65. Assume an interest rate of 8% compounded annually. By age 75, how much will each plan produce? Which plan would produce the larger amount?

61. Business Investment A firm of attorneys deposits $5000 of profit-sharing money at the end of each semiannual period for 7 1/2 years. Find the final amount in the account if the deposits earn 10% compounded semiannually. Find the amount of interest earned.

62. Business Financing A small resort must add a swimming pool to compete with a new resort built nearby. The pool will cost $28,000. The resort borrows the money and agrees to repay it with equal payments at the end of each quarter for 6 1/2 years at an interest rate of 12% compounded quarterly. Find the amount of each payment.

63. Business Financing The owner of Eastside Hallmark borrows $48,000 to expand the business. The money will be repaid in equal payments at the end of each year for 7 years. Interest is 10%. Find the amount of each payment.

64. Personal Finance To buy a new computer, Mark Nguyen borrows $3250 from a friend at 9% interest compounded annually for 4 years. Find the compound amount he must pay back at the end of the 4 years.

65. Effective Rate According to a financial Web site, on October 16, 2000, Guarantee Bank of Milwaukee, Wisconsin, paid 6.90% interest, compounded quarterly, on a 1-year CD, while Capital Crossing Bank of Boston, Massachusetts, paid 6.88% compounded monthly.† What are the effective rates for the two CDs, and which bank pays a higher effective rate?

66. Home Financing When the Lee family bought their home, they borrowed $115,700 at 10.5% compounded monthly for 25 years. If they make all 300 payments, repaying the loan on schedule, how much interest will they pay? (Assume the last payment is the same as the previous ones.)

67. New Car Chrysler’s “Summer Sales Drive” advertising campaign pledged a cash back allowance of $1500 or 0% financing for 60 months for a 2003 PT Cruiser.‡

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†www.bankrate.com.
Chapter 5 Mathematics of Finance

a. Determine the payments on a PT Cruiser if a person chooses the 0% financing option and needs to finance $17,000 for 60 months.

b. Determine the payments on a PT Cruiser if a person chooses the cash back option and now needs to finance only $15,500. Assume that the buyer is able to find financing from a local bank at 6.5% for 60 months compounded monthly.

c. Discuss which deal is best and why.

d. Find the interest rate at the bank that would make the other option optimal.

68. New Van Chrysler’s “Summer Sales Drive” advertising campaign pledged a cash back allowance of $3500 or 0% financing for 60 months for a 2003 Town and Country van.*

a. Determine the payments on a Town and Country van if a person chooses the 0% financing option and needs to finance $31,500 for 60 months.

b. If a person purchases a Town and Country van and chooses the cash back option, then she will need to finance $28,000. Assume that she is able to choose between two options at her local bank, 4.9% for 48 months or 5.5% for 60 months. Find the monthly payment and the total amount of money that she will pay back to the bank on each option.

c. Of the three deals, discuss which is the best and why.

69. Buying and Selling a House The Zambrano family bought a house for $91,000. They paid $20,000 down and took out a 30-year mortgage for the balance at 9%.

a. Find their monthly payment.

b. How much of the first payment is interest?

After 180 payments, the family sells its house for $136,000. They must pay closing costs of $3700 plus 2.5% of the sale price.

c. Estimate the current mortgage balance at the time of the sale using one of the methods from Example 4 in Section 3.

d. Find the total closing costs.

e. Find the amount of money they receive from the sale after paying off the mortgage.

The following exercise is from an actuarial examination.†

70. Death Benefit The proceeds of a $10,000 death benefit are left on deposit with an insurance company for 7 years at an annual effective interest rate of 5%. The balance at the end of 7 years is paid to the beneficiary in 120 equal monthly payments of $X, with the first payment made immediately. During the payout period, interest is credited at an annual effective interest rate of 3%. Calculate X.

a. 117  b. 118  c. 129  d. 135  e. 158

71. Investment The New York Times posed a scenario with two individuals, Sue and Joe, who each have $1200 a month to spend on housing and investing. Each takes out a mortgage for $140,000. Sue gets a 30-year mortgage at a rate of 6.625%. Joe gets a 15-year mortgage at a rate of 6.25%.

a. What annual interest rate, when compounded monthly, gives an effective annual rate of 10%?

b. What is Sue’s monthly payment?

c. If Sue invests the remainder of her $1200 each month, after the payment in part b, in a mutual fund with the interest rate in part a, how much money will she have in the fund at the end of 30 years?

d. What is Joe’s monthly payment?

e. You found in part d that Joe has nothing left to invest until his mortgage is paid off. If he then invests the entire $1200 monthly in a mutual fund with the interest rate in part a, how much money will he have at the end of 30 years (that is, after 15 years of paying the mortgage and 15 years of investing)?

f. Who is ahead at the end of the 30 years, and by how much?

g. Discuss to what extent the difference found in part f is due to the different interest rates or to the different amounts of time.

†Problem 16 from “Course 140 Examination, Mathematics of Compound Interest” of the Education and Examination Committee of The Society of Actuaries. Reprinted by permission of The Society of Actuaries.
EXTENDED APPLICATION: Time, Money, and Polynomials*

A time line is often helpful for evaluating complex investments. For example, suppose you buy a $1000 CD at time $t_0$. After one year $2500$ is added to the CD at $t_1$. By time $t_2$, after another year, your money has grown to $3851$ with interest. What rate of interest, called yield to maturity (YTM), did your money earn? A time line for this situation is shown in Figure 14.

**FIGURE 14**

Assuming interest is compounded annually at a rate $i$, and using the compound interest formula, gives the following description of the YTM.

\[ 1000(1 + i)^2 + 2500(1 + i) = 3851 \]

To determine the yield to maturity, we must solve this equation for $i$. Since the quantity $1 + i$ is repeated, let $x = 1 + i$ and first solve the second-degree (quadratic) polynomial equation for $x$.

\[ 1000x^2 + 2500x - 3851 = 0 \]

We can use the quadratic formula with $a = 1000$, $b = 2500$, and $c = -3851$.

\[ x = \frac{-2500 \pm \sqrt{2500^2 - 4(1000)(-3851)}}{2(1000)} \]

We get $x = 1.0767$ and $x = -3.5767$. Since $x = 1 + i$, the two values for $i$ are $0.0767 = 7.67\%$ and $-4.5767 = -457.67\%$. We reject the negative value because the final accumulation is greater than the sum of the deposits. In some applications, however, negative rates may be meaningful. By checking in the first equation, we see that the yield to maturity for the CD is $7.67\%$.

Now let us consider a more complex but realistic problem. Suppose Bill Poole has contributed for 4 years to a retirement fund. He contributed $6000$ at the beginning of the first year. At the beginning of the next 3 years, he contributed $5840$, $4000$, and $5200$, respectively. At the end of the fourth year, he had $29,912.38$ in his fund. The interest rate earned by the fund varied between $21\%$ and $-3\%$, so Poole would like to know the YTM = $i$ for his hard-earned retirement dollars. From a time line (see Figure 15), we set up the following equation in $1 + i$ for Poole’s savings program.

\[ 6000(1 + i)^4 + 5840(1 + i)^3 + 4000(1 + i)^2 + 5200(1 + i) = 29,912.38 \]

Let $x = 1 + i$. We need to solve the fourth-degree polynomial equation

\[ f(x) = 6000x^4 + 5840x^3 + 4000x^2 + 5200x - 29,912.38 = 0. \]

There is no simple way to solve a fourth-degree polynomial equation, so we will use a graphing calculator.

We expect that $0 < i < 1$, so that $1 < x < 2$. Let us calculate $f(1)$ and $f(2)$. If there is a change of sign, we will know that there is a solution to $f(x) = 0$ between 1 and 2. We find that

\[ f(1) = -8872.38 \quad \text{and} \quad f(2) = 139,207.62. \]

Using a graphing calculator, we find that there is one positive solution to this equation, $x = 1.14$, so $i = \text{YTM} = .14 = 14\%$.

**Exercises**

1. Brenda Bravener received $50 on her 16th birthday, and $70 on her 17th birthday, both of which she immediately invested in the bank with interest compounded annually. On her 18th birthday, she had $127.40 in her account. Draw a time line, set up a polynomial equation, and calculate the YTM.

2. At the beginning of the year, Jay Beckenstein invested $10,000 at $5\%$ for the first year. At the beginning of the second year, he added $12,000 to the account. The total account earned $4.5\%$ for the second year.

   a. Draw a time line for this investment.
   b. How much was in the fund at the end of the second year?
   c. Set up and solve a polynomial equation and determine the YTM. What do you notice about the YTM?

3. On January 2 each year for 3 years, Greg Odjakjian deposited bonuses of $1025$, $2200$, and $1850$, respectively, in an account. He received no bonus the following year.

5. People often lose money on investments. Jim Carlson invested $50 at the beginning of each of 2 years in a mutual fund, and at the end of 2 years his investment was worth $90.
   a. Draw a time line and set up a polynomial equation in $1 + i$. Solve for $i$.
   b. Examine each negative solution (rate of return on the investment) to see if it has a reasonable interpretation in the context of the problem. To do this, use the compound interest formula on each value of $i$ to trace each $50 payment to maturity.

**Directions for Group Project**

Assume that you are in charge of a group of financial analysts and that you have been asked by the broker at your firm to develop a time line for each of the people listed in the exercises above. Prepare a report for each client that presents the YTM for each investment strategy. Make sure that you describe the methods used to determine the YTM in a manner that the average client should understand.