

Worst-case to Average-case Hardness of LWE: An Alternative Perspective



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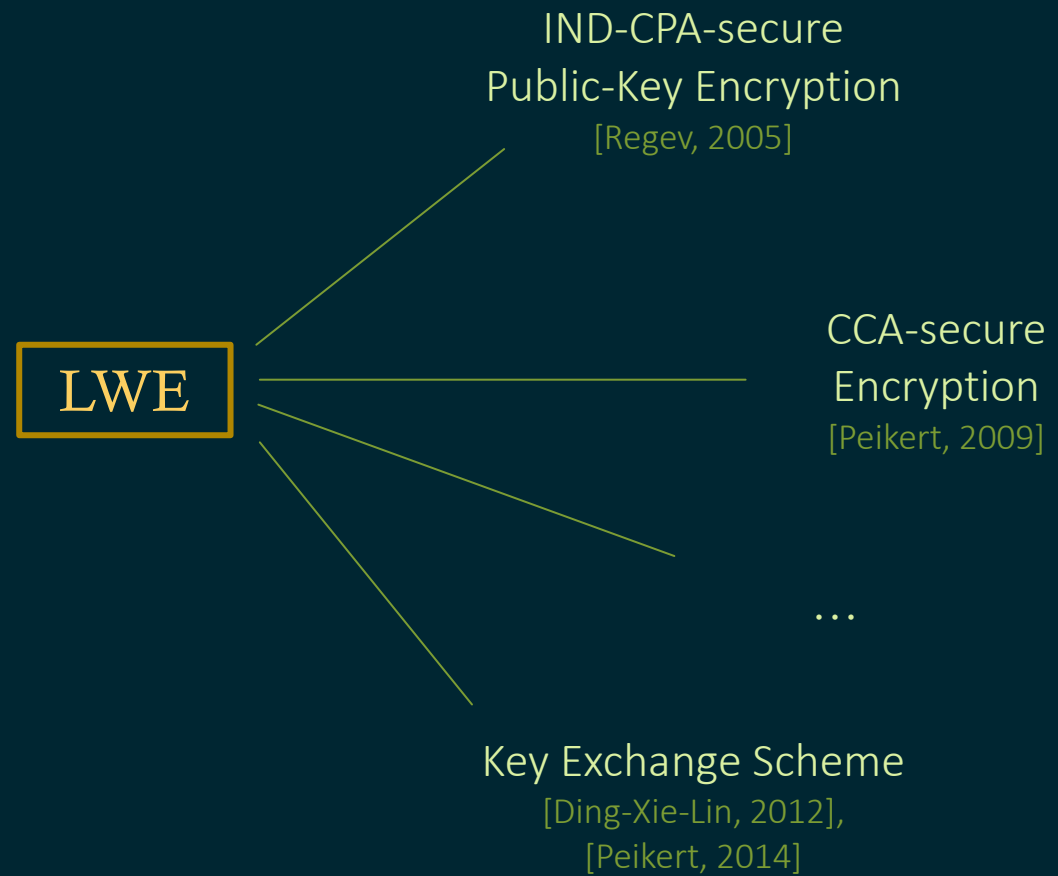


Leong Jin Ming

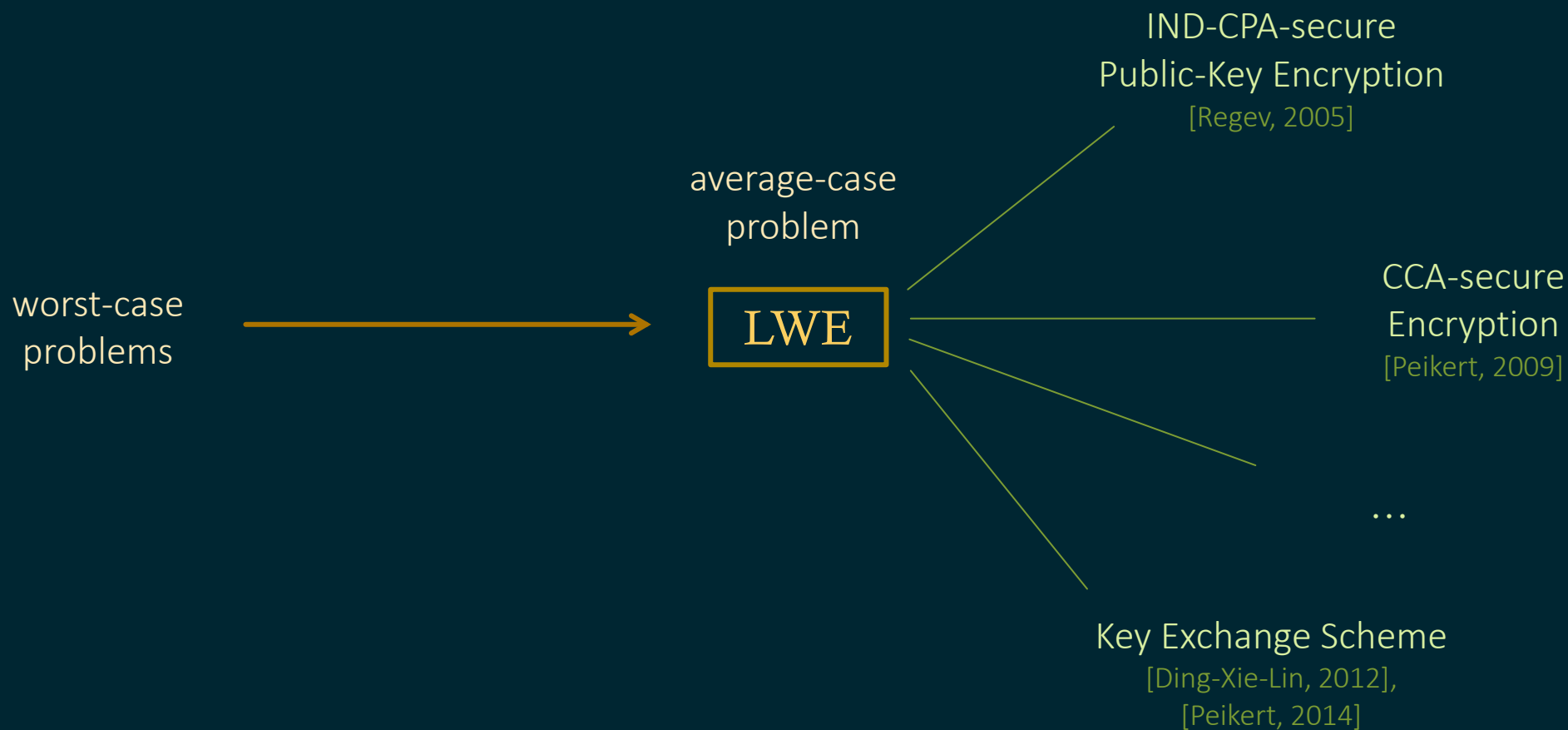


Alexandra Veliche

Cryptographic from LWE



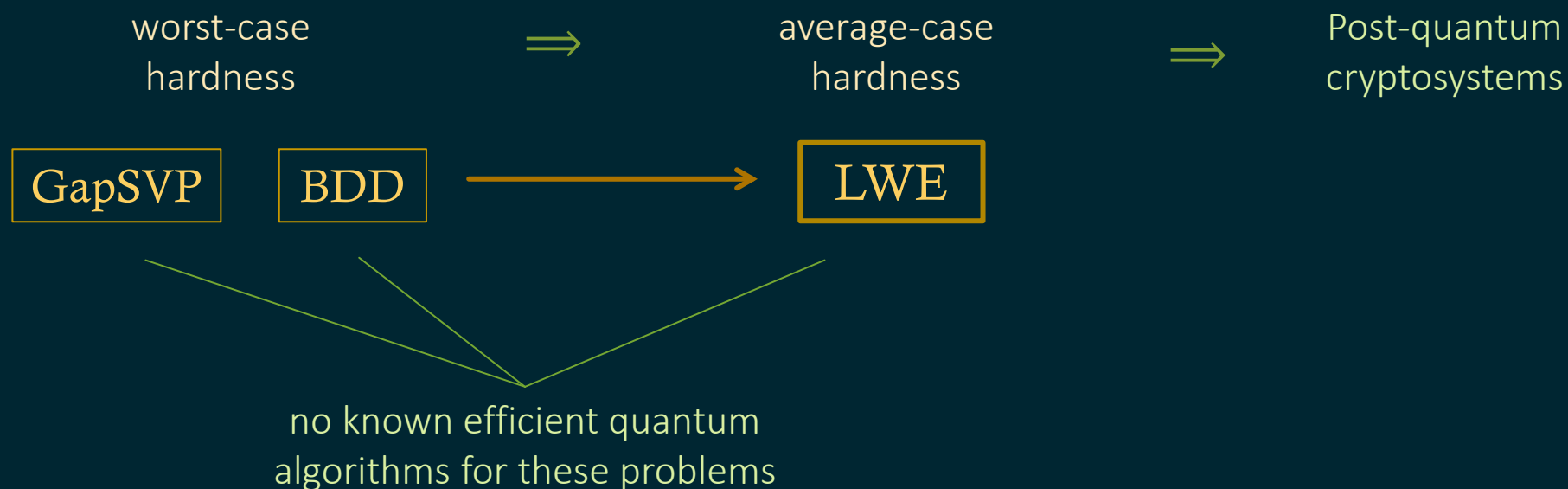
Cryptographic Significance



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Learning With Errors

LWE_{n,p,φ} : n dimension, p modulus, $\phi \sim \mathbb{R}/\mathbb{Z}$ error distribution

Given noisy samples $(\mathbf{a}, \langle \mathbf{a}, \mathbf{s} \rangle + e)$, where

$\mathbf{a} \leftarrow \mathbb{Z}_p^n$ uniformly random, $\mathbf{s} \in \mathbb{Z}_p^n$ unknown, $e \leftarrow \phi$ small error,

(search-LWE) output $\mathbf{s}$.

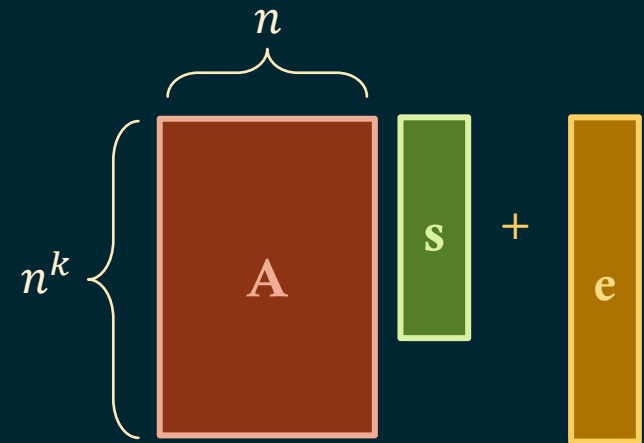
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Learning With Errors

LWE $_{n,p,\phi}$: n dimension, p modulus, $\phi \sim \mathbb{R}/\mathbb{Z}$ error distribution

Given noisy samples $(\mathbf{a}, b)$, where

$\mathbf{a} \leftarrow \mathbb{Z}_p^n$ uniformly random and $b \in \mathbb{Z}_p$,

(*decision-LWE*) output

- YES if samples are from the LWE distribution for $\mathbf{s}$ and ϕ ,
- NO if samples are uniformly random.

Lattices

Lattice:

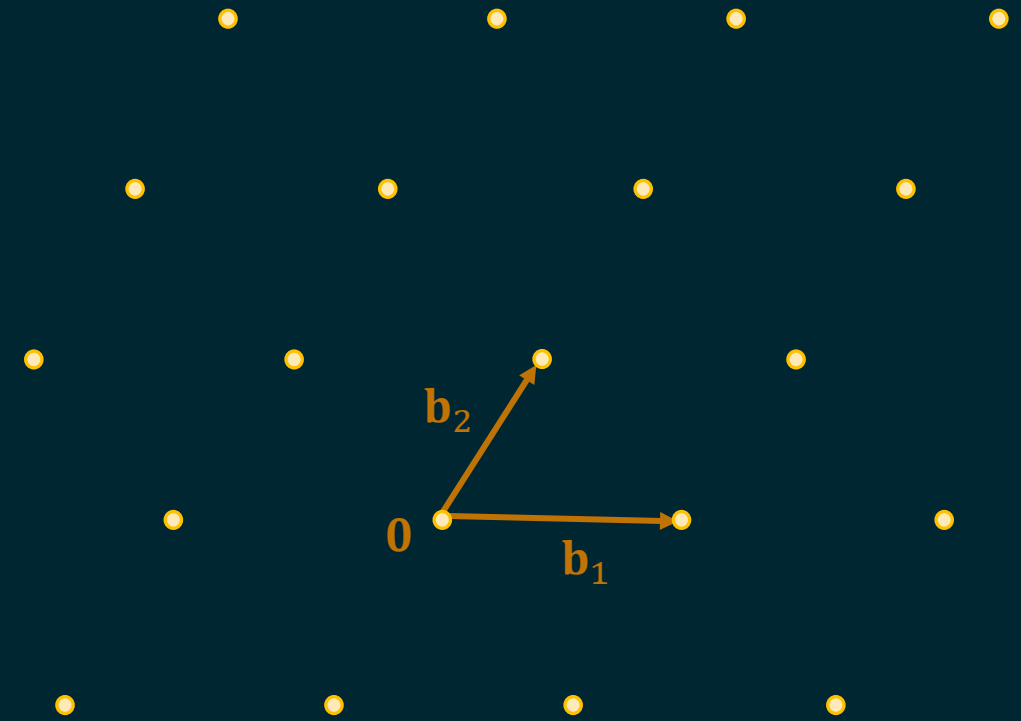
An infinite discrete set of vectors in $\mathbb{R}^n$

consisting of all integer linear combinations

$$\mathcal{L} = \{a_1 \mathbf{b}_1 + \cdots + a_k \mathbf{b}_k : a_1, \dots, a_k \in \mathbb{Z}\}$$

of some linearly independent vectors $\mathbf{b}_1, \dots, \mathbf{b}_k \in \mathbb{R}^n$.

The set $\{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ is called a *basis*.



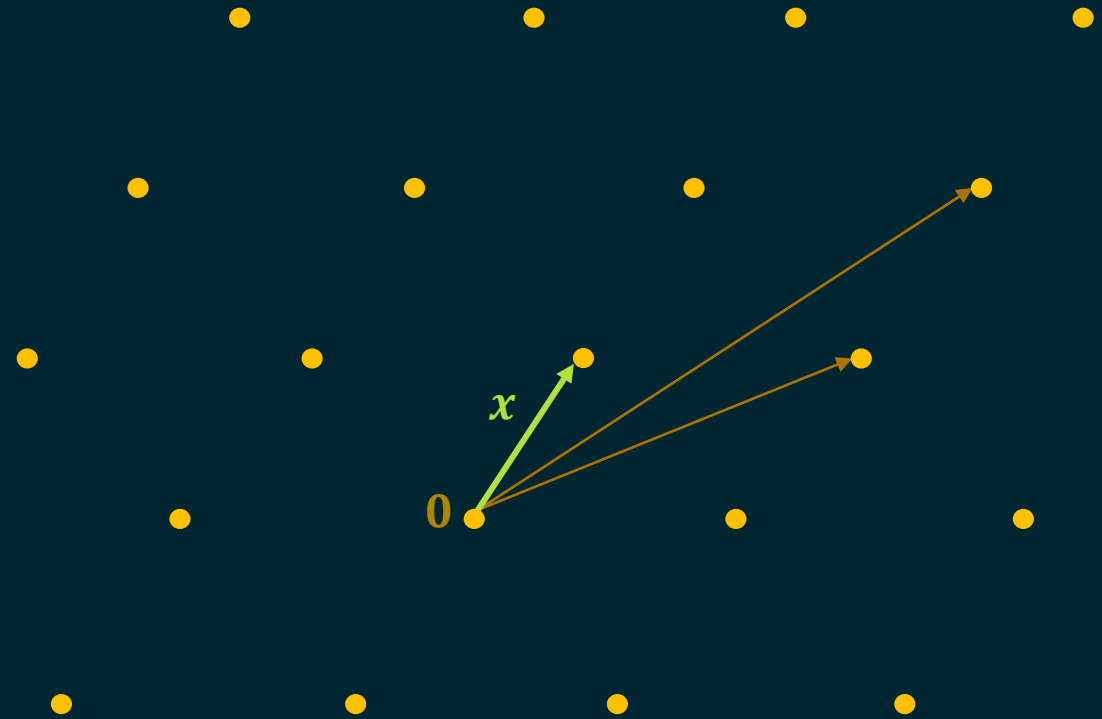
Shortest Vector Problem

SVP :

Given a basis $\mathcal{B}$ for lattice $\mathcal{L} \subset \mathbb{R}^n$,

find a shortest non-zero lattice vector $\mathbf{x}$,

i.e. $\mathbf{x} \in \mathcal{L} \setminus \{\mathbf{0}\}$, such that $\|\mathbf{x}\| = \lambda_1(\mathcal{L})$.



Shortest Vector Problem

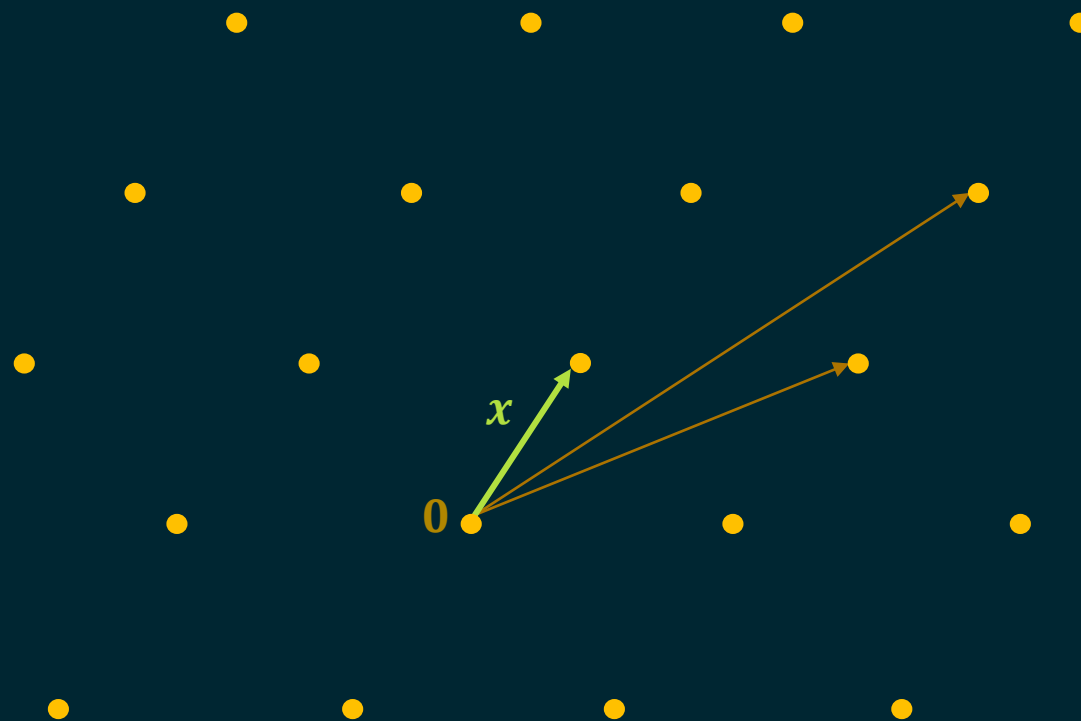
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GapSVP _{γ} is an approximate decision variant.



Bounded Distance Decoding

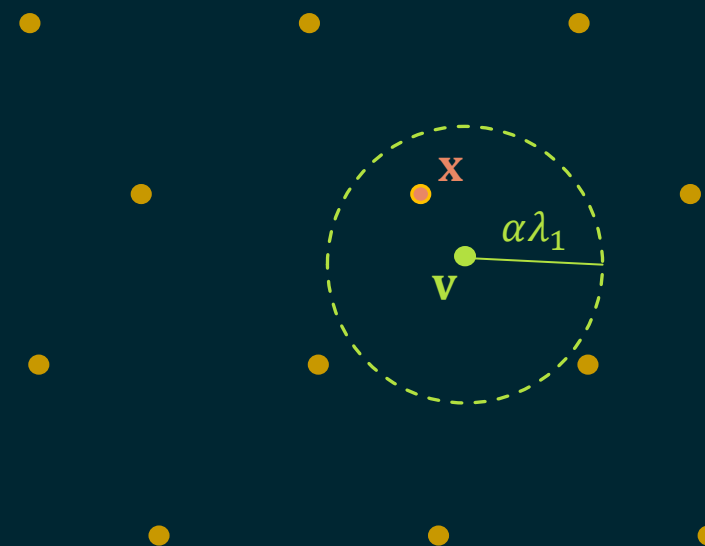
BDD_α : $\alpha > 0$ distance approximation factor

Given a basis $\mathcal{B}$ for a full-rank lattice $\mathcal{L} \subseteq \mathbb{R}^n$

and a target vector $\mathbf{v} \in \mathbb{R}^n$ close to the lattice,

find a lattice vector $\mathbf{x} \in \mathcal{L}$ closest to $\mathbf{v}$,

i.e. $\mathbf{x} \in \mathcal{L}$ such that $\|\mathbf{v} - \mathbf{x}\|_2 < \alpha \cdot \lambda_1(\mathcal{L})$.



Hardness of LWE

[Regev, 2009] — quantum reduction from worst-case lattice problems to decision-LWE



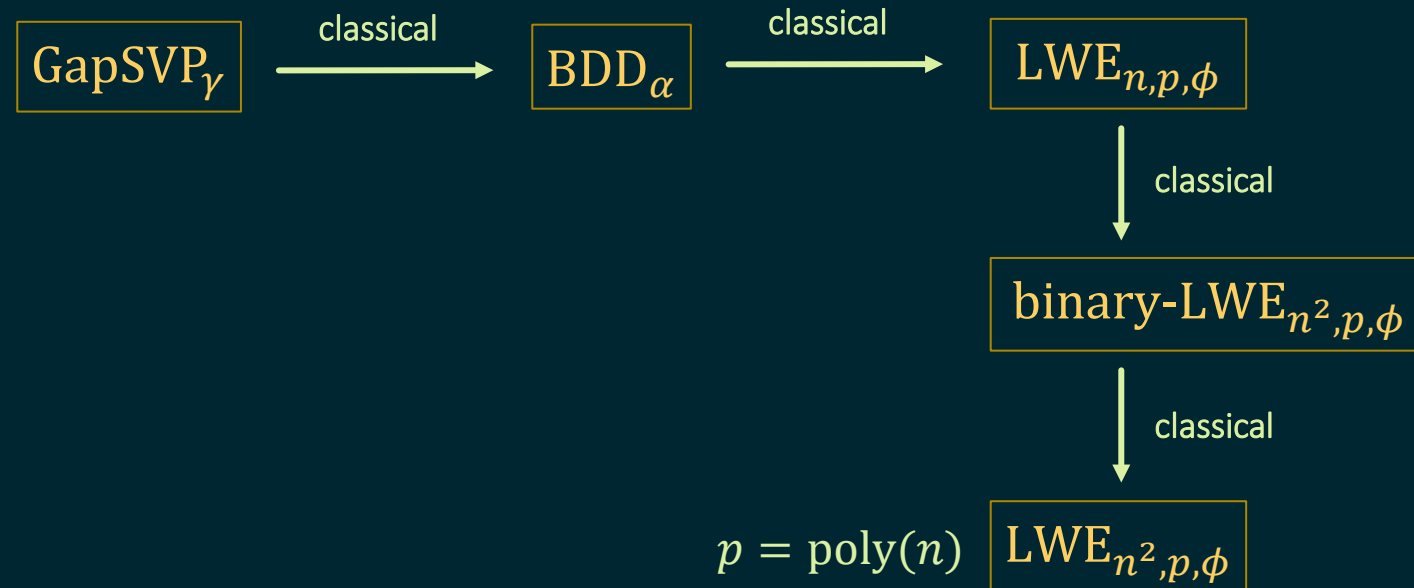
Hardness of LWE

[Peikert, 2009] — classical reduction, but modulus becomes *exponential*

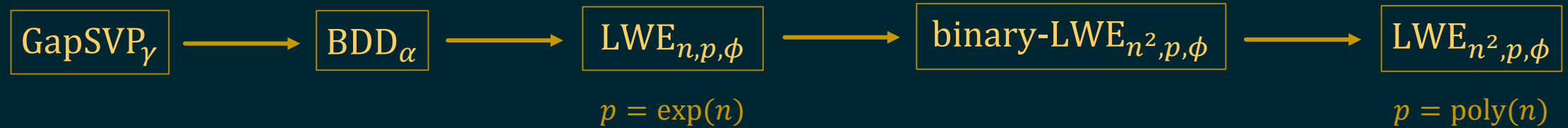


Hardness of LWE

[Brakerski, Peikert, Langlois, Regev, Stehle, 2013] — classical reduction with polynomial modulus



Hardness of LWE



Algorithms for Lattice Problems



Fastest algorithms for these problems run in $2^{\Theta(n)}$ time (for polynomial approximation factor).

What the Reduction says about LWE Algorithms

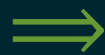


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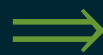
LWE cannot be solved in less
than $2^{\Omega(\sqrt{n})}$ time.

What the Reduction says about LWE Algorithms

[Blum-Kalai-Wasserman, 2000] — Best known algorithm for $\text{LWE}_{n,p,\phi}$ runs in $2^{O(n \log p / \log n)}$ time.



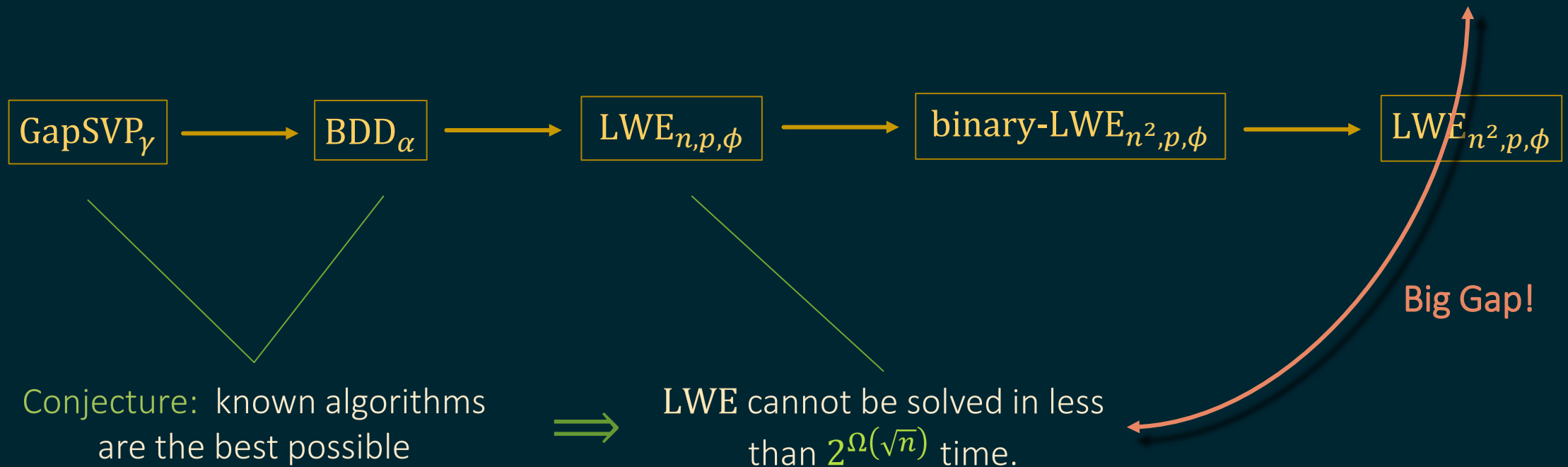
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Closing the Gap

How to close this gap?

We change our perspective!

Security in Practice

What does it mean for a cryptosystem to be 256-bit secure?



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- (a) The fastest algorithm for breaking the cryptosystem runs in 2^{256} time.
- (b) No reasonably efficient algorithm can break the cryptosystem with probability $> 2^{-256}$.



Security in Practice

What does it mean for a cryptosystem to be 256-bit secure?

(a) The fastest algorithm for breaking the cryptosystem runs in 2^{256} time.

(b) No reasonably efficient algorithm can break the cryptosystem with probability $> 2^{-256}$.

This is what we usually want
for cryptographic security



An Alternative Perspective

An alternative measure of computational hardness:

The maximum success probability of any PPT algorithm that finds a solution.



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Can we study worst-case to average-case reductions under this framework?

Yes (this talk!)

Success Probability of Solving LWE

Trivial algorithm (guess the error): Success probability for solving $\text{LWE}_{n,p,\phi}$ is $p^{-\Omega(n)}$.

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All other algorithms are not PPT, so it is unlikely that we can achieve better than this.

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LLL / Slide Reduction + guess coefficients: Success probability of solving GapSVP_γ is $2^{-\Theta(n^2/\log n)}$.

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BDD_α is closely related to GapSVP_γ for $\gamma = \text{poly}(n) = \frac{1}{\alpha}$,
so it is unlikely we can achieve better than known algorithms.

A Natural Conjecture

Conjecture:

(informal) No algorithm can solve BDD_α on an arbitrary n -rank lattice for $\alpha = 1/\text{poly}(n)$ in polynomial time with success probability better than $2^{-n^2/\log n}$.

What We Show

Trivial algorithm: Success probability for efficiently solving $\text{LWE}_{n,p,\phi}$ is $p^{-\Omega(n)}$.

Conjecture $\implies$ Maximum success probability for efficiently solving $\text{LWE}_{n,p,\phi}$ is $p^{-\Omega(n/\log^2 n)}$.

What We Show

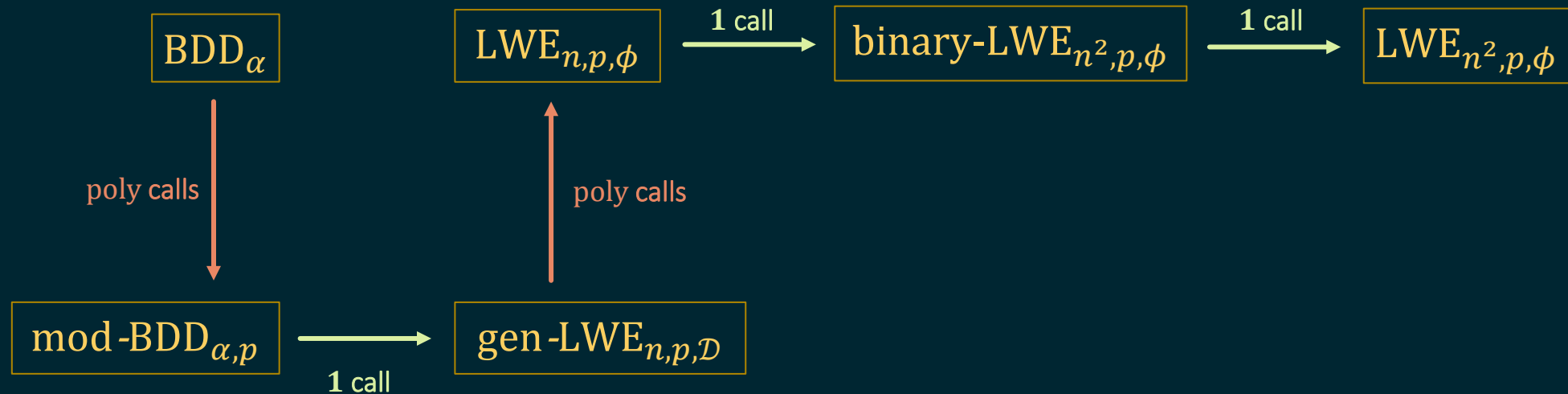
Trivial algorithm: Success probability for efficiently solving $\text{LWE}_{n,p,\phi}$ is $p^{-\Omega(n)}$.

Tight!

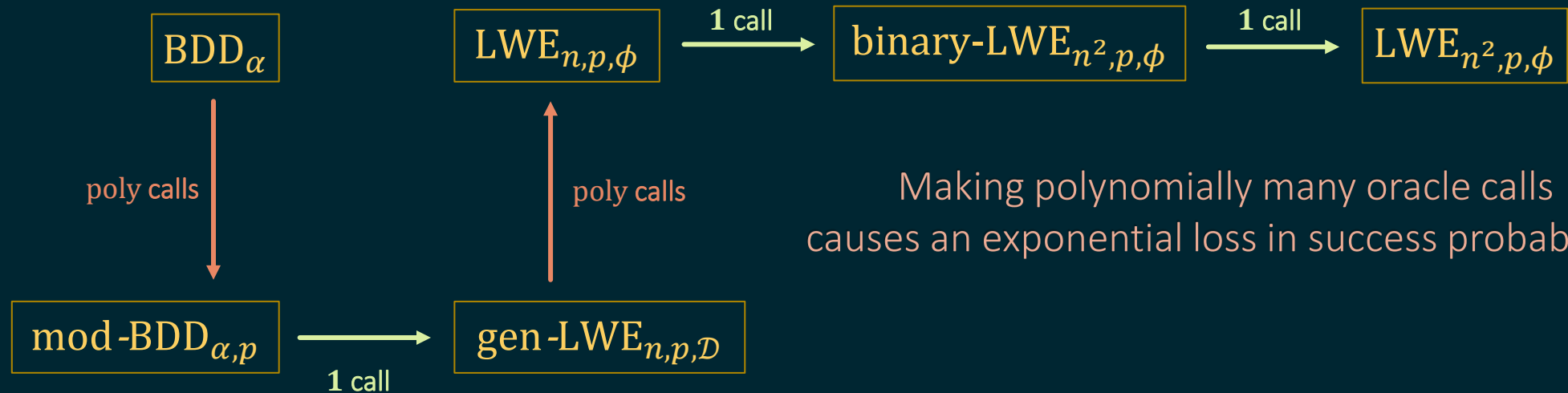


Conjecture $\implies$ Maximum success probability for efficiently solving $\text{LWE}_{n,p,\phi}$ is $p^{-\Omega(n/\log^2 n)}$.

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Making polynomially many oracle calls causes an exponential loss in success probability!

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Reduction algorithm for $\mathcal{P} \rightarrow \mathcal{Q}$ makes k calls to oracle for $\mathcal{Q}$.

Success probability of solving $\mathcal{Q}$ is $\geq \epsilon \implies$ success probability of solving $\mathcal{P}$ is $\geq \epsilon^k$.

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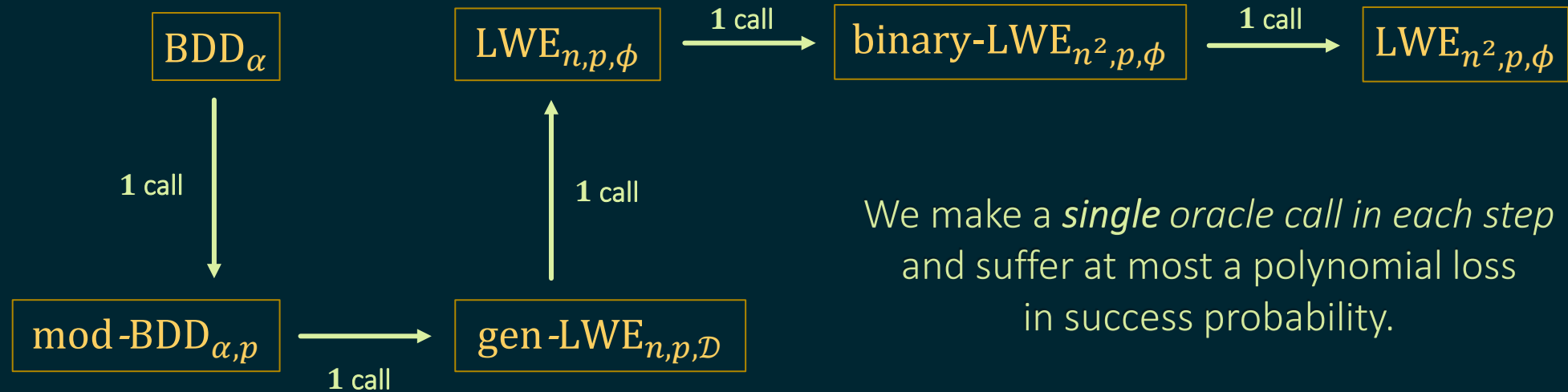
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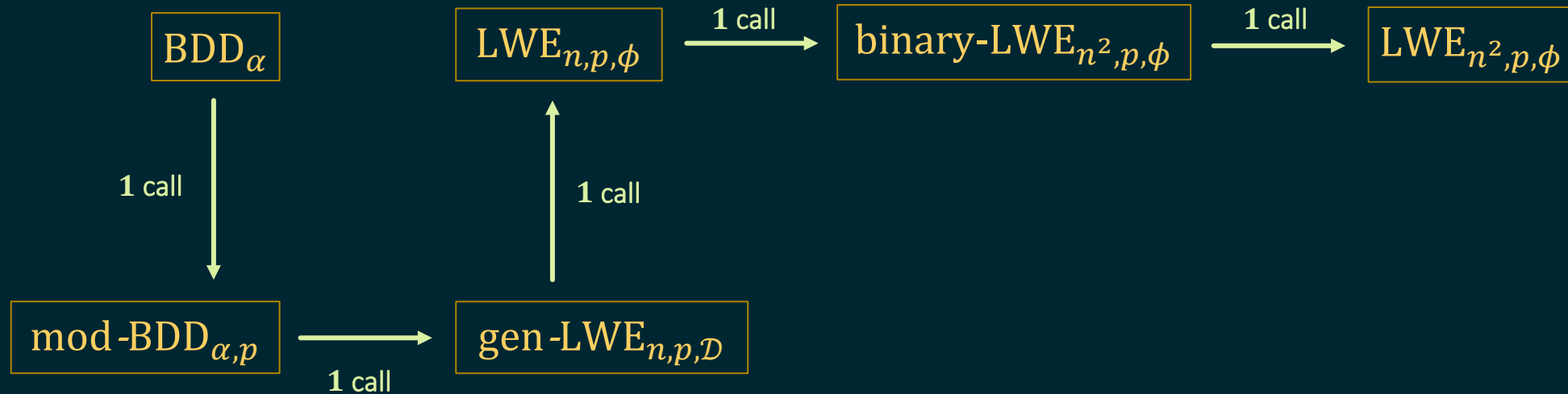
Success probability of solving $\mathcal{P}$ is $\leq \delta \implies$ success probability of solving $\mathcal{Q}$ is $\leq \delta^{1/k}$.

We want just $O(1)$ oracle calls to get a meaningful conclusion.

Our Reduction



Our Reduction



We use the same techniques as [Regev, 2005] and [Brakerski+, 2013], but with great care to the *explicit loss in success probability* and *number of oracle calls*.

Our Main Result

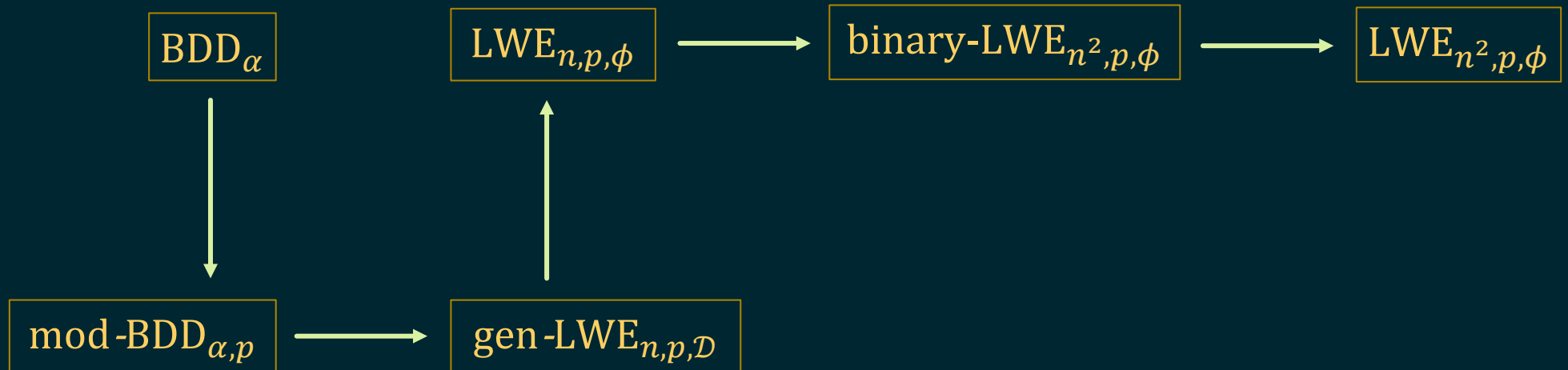
Theorem 1: (*informal*) If no PPT algorithm can solve BDD_α for $\alpha \in (0, 1/2)$

with success probability greater than $2^{-\Omega\left(\frac{n^2}{\log n}\right)}$,

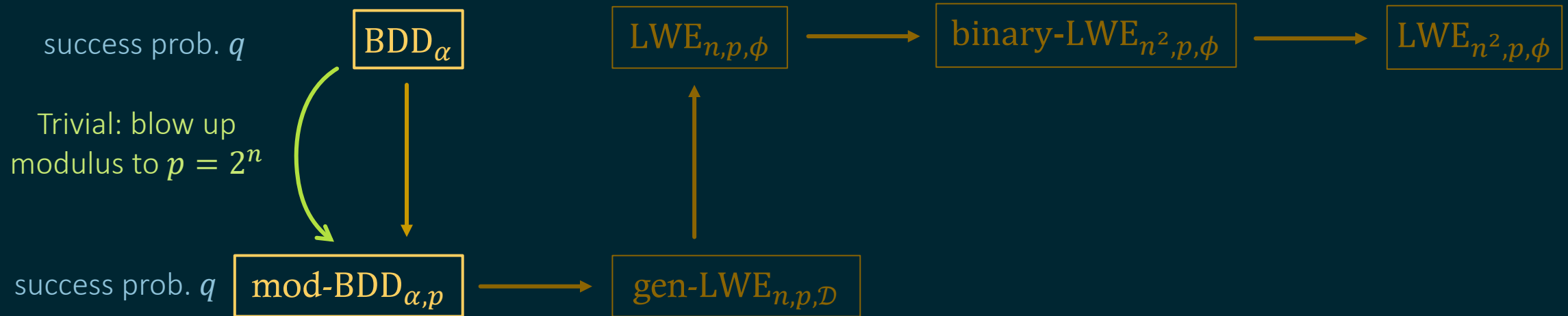
then no PPT algorithm can solve $\text{search-LWE}_{n,p,\phi}$ (even for binary secret)

for dimension n , and modulus $p = \text{poly}(n)$ with success probability $2^{-\frac{n}{\log n}}$.

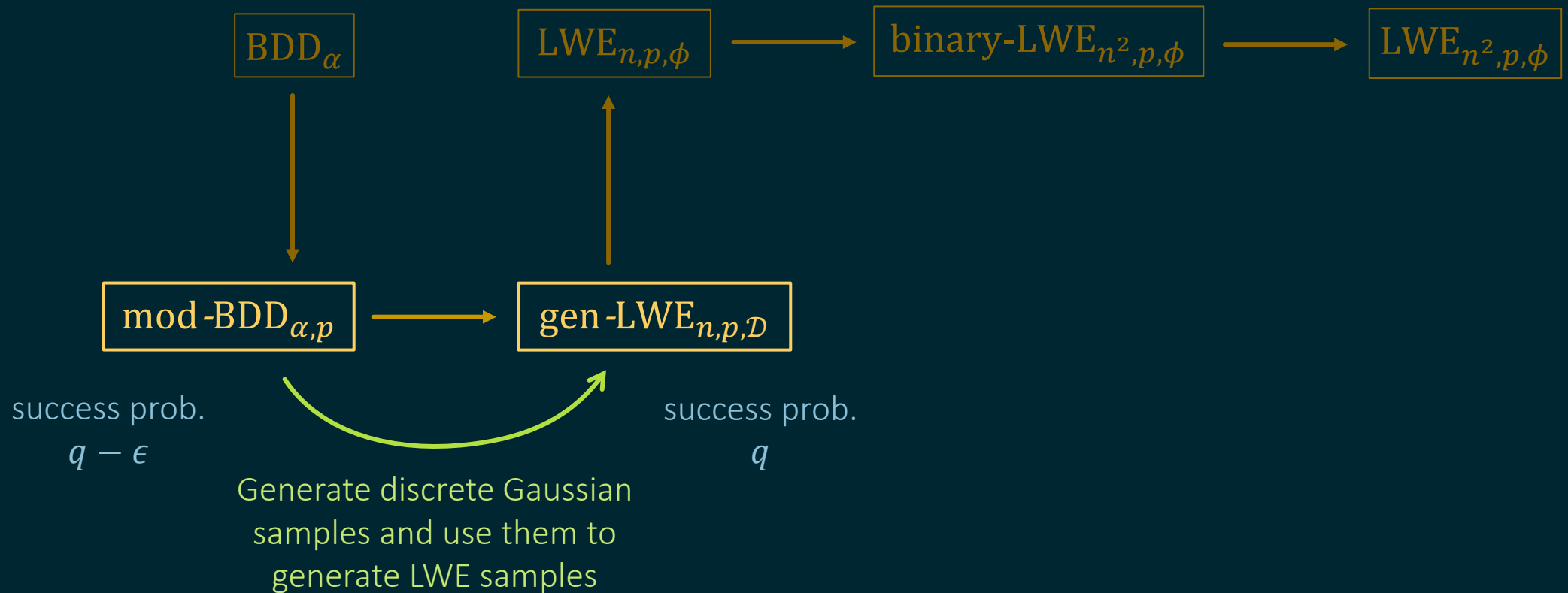
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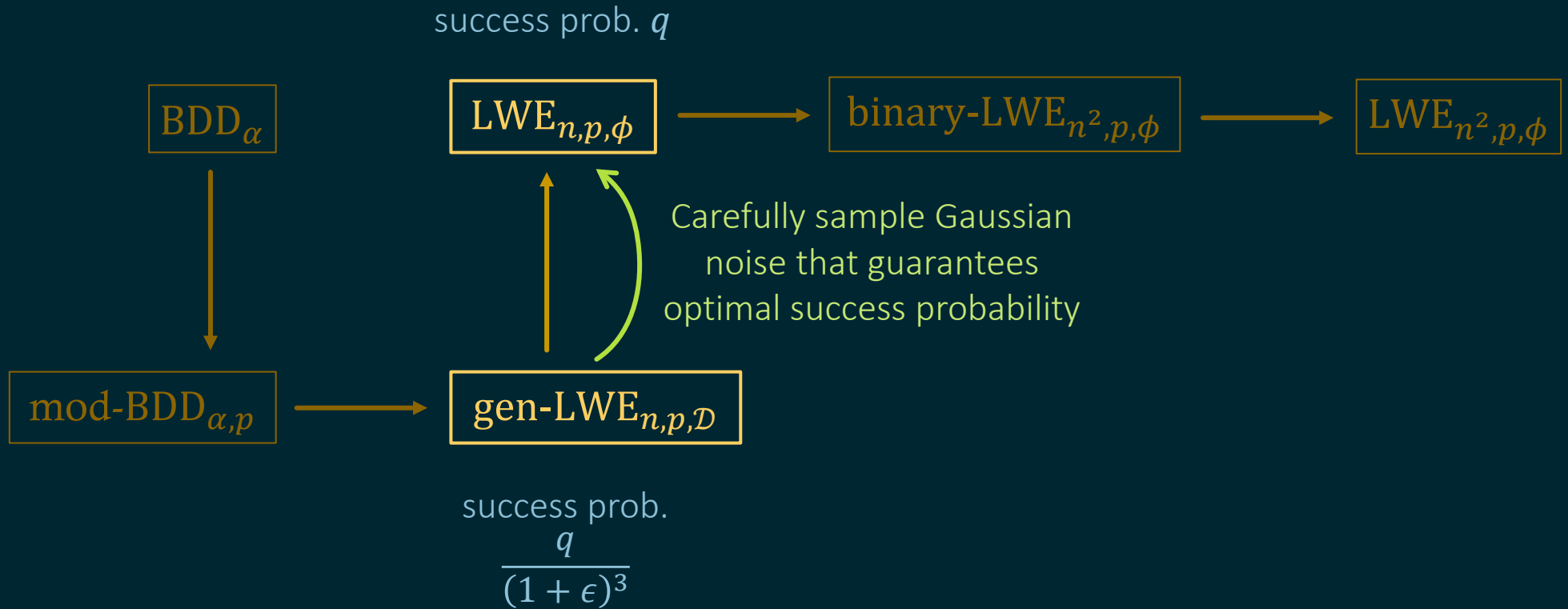
Our Proof Techniques



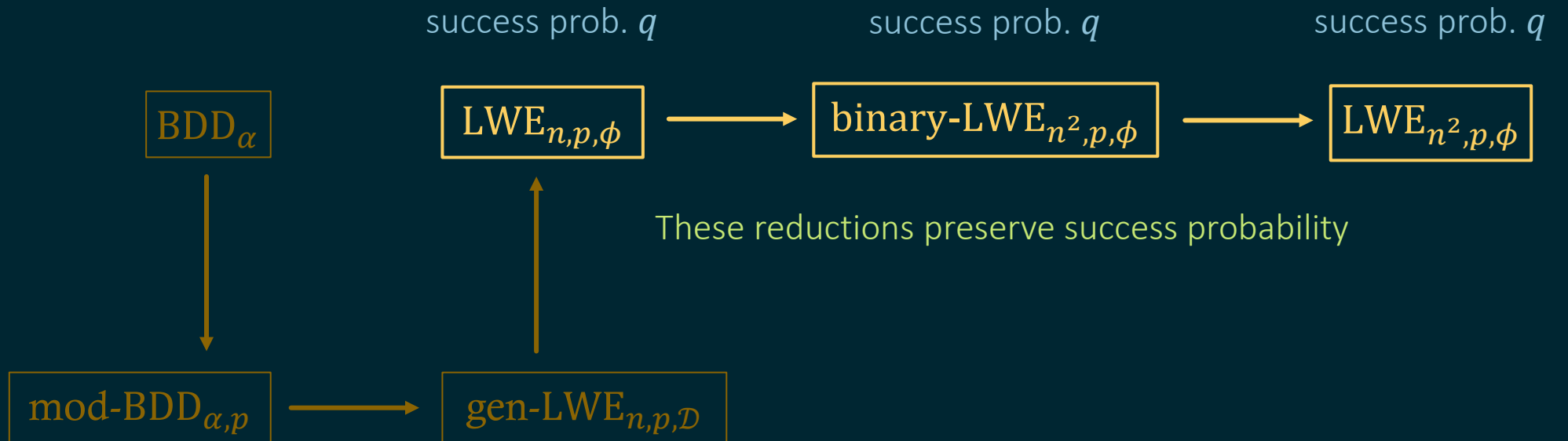
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Our Second Result

Theorem 2: (*informal*) If no algorithm can solve **search-LWE** _{n,p} for polynomial modulus with success probability α in *expected* polynomial time, then no PPT algorithm can “solve” **decision-LWE** _{n,p} with probability $\approx \alpha$.

Future Directions

- Establish a similar result for **GapSVP** $\rightarrow$ **BDD** (or prove impossibility).
- Reductions **BDD** $\rightarrow$ search-LWE and search-LWE $\rightarrow$ decision-LWE are disconnected, because *expected* polynomial-time is a fundamental part of the second reduction.
Is a workaround possible?
- Use this alternative framework to study the complexity of other computational problems relevant to cryptography or learning.

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Thank you! Questions?