

# An Experimental Demonstration of the Viability of Energy Minimizing Computing using Nano-magnets

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**Abstract**—There has been recent proposals for the use of nano-magnets to directly solve quadratic minimization problems, especially those arising in computer vision applications. This is unlike proposals for using nano-magnets to represent binary states. A collection of nano-magnets, when driven to their ground states, can be seen to optimize a quadratic energy function that is determined by their relative placement. By controlling the relative placement of nano-magnets, we can change the energy function being minimized. In this work, we experimentally demonstrate this capability by fabricating and testing an example of a quadratic optimization problem that accomplishes line grouping.

## I. INTRODUCTION

There has been significant growth in interest in field coupled computing as a radically different computing paradigm. One architecture that has been proposed is the Cellular Automata architecture [1], [2]. Cellular automata architectures based on Coulombic interactions of electrons (Quantum-dot cellular automata, QCA) have been proposed [3], [4], [5], [6], [3]. Another possibility is to use magnetic coupling. This is particularly attractive since this form of computing can be done at room temperatures [7], unlike those based on electronic charge interactions, which require very low temperatures. Bandhyopadhyay [8] has advocated the use of local spin coupling for computing, a middle ground between quantum computing and spinotronic transistors. The architecture is essentially a cellular automata architecture, but with spin coupling energies for logic computing.

So far, suggestions for field coupled computing has been for Boolean logic based computing [7], [2]. Indeed, most work on nano-logic that seeks to replicate traditional computing involving logic and arithmetic operations [9], [10], [11]. Other applications that have been proposed with QCAs are signal processing [12], permutation matrices [13], interconnection networks [14], and fast Fourier transforms [15]. *However, the energy minimization aspects of CAs have not been directly harnessed* [16].

Since nano-devices are expected to have high error rates, both fabrication related and during operations, it makes sense to consider *error-tolerant applications* where the cost of failure of not finding *the* optimal solution is not high; even

solutions that are close to optimal ones suffices in practice. One such context is in quadratic optimization that arises in computer vision. Energy minimization plays a central role in computer vision algorithms. Nano-magnets offer a tantalizing alternative to traditional form of digital computing for solving quadratic energy minimization problems, drastically reducing the computational time required. Collection of nano-magnets, when driven to their ground states, can be seen to optimize a quadratic energy function that is determined by their relative placement. By controlling the relative placement of magnets, we can change the energy function being minimized. It is somewhat like analog computing from the past, except that instead of solving differential equations we solve minimization problems.

Imagine an energy minimization co-processor based on nano-magnets that is heterogeneously integrated with CMOS. In the long term, these vision computing circuits can be integrated with camera circuitry to design cognitive cameras, capable of higher level reasoning. The "complier" for this form of computing would transform a given energy minimization problem into a set of equivalent coordinates for a magnet collection. These magnets would be "selected" from a regular grid of nano-magnets by driving the non-computing magnets into non-interacting vortex states. The array would then be clocked to its ground state. The final states would be read off as the solution to the problem. The vision problem would then use these magnetic measurements as the solution. We will essentially be solving an optimization problem with each input-and-readout cycle as compared to orders of magnitude more clock cycles that would be needed in a Boolean logic circuit. The current work is towards this long-term goal. We experimentally demonstrate the viability of using single domain nano-magnetic coupling for function minimization computing.

Unlike logic and arithmetic computing tasks that demand exact computations, vision problems can work with near optimal solutions. These vision problems place high demand on computational resources (on Boolean logic based computing platforms). There have been proposals for using regular arrays of quantum-dots [17], [18] and nano-magnets [19] for low-level vision, mainly segmentation where the input and the output are both regular grid of pixels. In this work we consider quadratic energy forms that arise in contexts involving extended image features rather than individual pixels. These problems have high computational complexity and are not amenable for single-instruction-multiple-data (SIMD) type hardware solutions. To solve the vision problems that we target, VLSI implementation using traditional

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logic would require complex MIMD architectures as opposed to just SIMD architectures, which are prevalent in the design of vision chips [20]. For instance, a recent proposal for an object recognition chip uses both SIMD and MIMD components [21].

## II. VISION BACKGROUND

The term perceptual organization is used to describe the act of recognizing important features of an image. For instance in Figure 2 (a), the task is to find the visually salient or important edges, such as the three parallel lines in the center of the image. The detection or grouping of significant features in an image is a computationally expensive process. A useful fact is that grouping does not require exact computational accuracy, meaning that solutions that are near the optimal result are acceptable [22], [23]. The grouping of low-level segments can be accomplished through a quadratic energy minimization process [19]. Let there be  $N$  straight lines in the image that we would like to group. With each straight line, let us associate a variable,  $x_i$ , taking on values 1 or 0, denoting whether it is significant or not significant, respectively. Every pair of straight lines can be associated with an affinity value capturing its saliency (or perceptual importance). Various functional forms have been proposed in the computer vision literature for this affinity function. They all are designed to capture perceptual organization of the straight lines. For instance, if two straight lines are parallel to each other they will have high affinity than any other random arrangements. The justification for this being that it is highly unlikely for lines to be parallel to each other by chance. There must be an underlying reason, and they are very likely belong to some object in the scene. One mathematical form that captures the pairwise saliency of the  $i$ -th and  $j$ -th line segment in the image is the following

$$a_{ij} = \sqrt{l_i l_j} \exp^{-\frac{o_{ij}}{\max l_i, l_j}} \exp^{-\frac{d_{min}}{\max l_i, l_j}} \cos^2(2\theta_{ij}) \quad (1)$$

where  $l_i$  and  $l_j$  are the lengths of the two segments,  $\theta_{ij}$ , is the angle between them,  $o_{ij}$  is the overlap with each other, and  $d_{min}$  is the minimum distance between them. As one can see from the expression, affinity between two straight lines will be high for longer segments, segments that are parallel to each other, or continuous to each other, or at right angles to each other. Given these pairwise affinity values, the vision problem is to find the values of  $x_i$  for each segment such that the following measure is maximized.

$$\sum_i \sum_{j \neq i} a_{ij} x_i x_j + (k - \sum_i x_i) \quad (2)$$

The first term is the total of the pairwise affinities among the segments with  $x_i = 1$  and the second term tries to enforce that we have  $k$  segments with  $x_i = 1$ . This is a hard problem to solve. Traditional Boolean logic based approaches would reduce this binary quadratic problem, into finding exact solutions based on arithmetic and logical operations. This is very demanding on a Boolean system and is not necessarily required for the grouping of visual objects and magnetic

nano-systems present a unique way to accomplish such a task in a direct manner.

## III. MAGNETIC HAMILTONIAN

The basic unit of computation for magnetic logic is a nano-magnet with dimensions and materials such it exhibits single domain behavior, i.e. it can be modeled as one overall magnetic state. The material composition and the geometry (shape and size) of the nano-magnet determine the overall magnetic behavior. For instance, for disk-shaped magnets that are thin (say 20 nanometers, nm) and with diameter of 100 nm, all the magnetic vectors are aligned perpendicular to the  $z$ -direction, in the  $xy$ -plane (single domain). The vectors are either all aligned one direction, resulting in one overall effective magnetic vector direction, or aligned in a circular fashion, resulting in a vortex state. For a vortex state, there is only a small effective magnetic vector in the  $z$ -direction at the center of the magnet, but the overall magnetic effect in the  $xy$ -direction is zero. Let the flat nano-scale disks be of height  $h$ , radius  $r$ , and magnetization  $M_0$  ordered in an array in the  $xy$ -plane. Bennett and Xu [24] showed that a disk of uniform magnetization can be approximated well by a point dipole with moment  $\pi r^2 h M_0$  that is oriented in the plane forming an angle  $\phi$  with the  $x$ -axis and with  $m(z) = 0$ . The magnetization vector of the  $i$ -th magnet can be represented by  $\mathbf{m}_i$ . The total Hamiltonian of an arrangement of magnets is given by:

$$\mathcal{H} = \sum_i \mathbf{m}_i^T \mathbf{D}_i \mathbf{m}_i + \mu_0 \sum_i \mathbf{m}_i^T \mathbf{h}_{\text{ext}} + \sum_i \sum_{j \neq i} \mathbf{m}_i^T \mathbf{C}_{ij} \mathbf{m}_j \quad (3)$$

where  $\mathbf{D}_i$  is the demagnetization tensor of the  $i$ -th magnet, capturing the shape anisotropy,  $\mathbf{C}_{ij}$  is the interaction matrix between the  $i$ -th and  $j$ -th magnet,  $\mathbf{h}_{\text{ext}}$  is the external field. A diagonal matrix with value of 1/2 along the diagonal can approximate the demagnetization tensor for a thin disk, so the first energy term is a constant.

$$\mathcal{H} = D_0 + \mu_0 \sum_i \mathbf{m}_i^T \mathbf{h}_{\text{ext}} + \sum_i \sum_{j \neq i} \mathbf{m}_i^T \mathbf{C}_{ij} \mathbf{m}_j \quad (4)$$

For dipole to dipole interaction approximation, the coupling term (the third term) is given by

$$\mathbf{C}_{ij} = \frac{\mu_0 |M|}{4\pi d_{ij}^3} (3\mathbf{e}_{ij} \mathbf{e}_{ij}^T - \mathbf{I}) \quad (5)$$

$$= c_{ij} (3\mathbf{e}_{ij} \mathbf{e}_{ij}^T - \mathbf{I}) \quad (6)$$

where  $\mathbf{e}_{ij}$  is the *unit* vector line joining the centers of the two dipoles,  $d_{ij}$  is the distance between the centers, and  $\mathbf{I}$  is the identity matrix [25]. The term  $|M|$  is the product of the magnetic moment magnitudes of the two magnets and is constant for our magnets. The interaction term between  $i$ -th and  $j$ -th magnet will be dependent on the relative placement of the magnets. For any particular magnet if the interaction was low with the other magnets, then it would be easy to change its magnetization vector with a low external field. Conversely, if the interaction with other magnets were high then it would be hard to change its magnetization using a low external field.

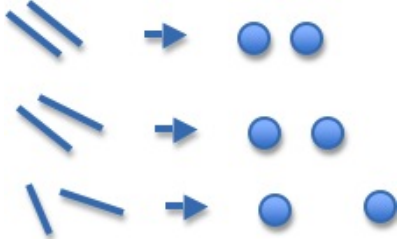


Fig. 1. Correspondence between nano-magnets (MFC) and vision problem formulation. On the left are sample arrangements of pairs of straight lines and on the right are the corresponding nano-magnet placements whose interactions match the affinities between the lines.

#### IV. CORRESPONDENCE BETWEEN VISION AND MAGNETS

Notice the correspondence between Equations 2 and 4. Each line segment in the image correspond to a magnet. The magnetizations,  $\mathbf{m}_i$ , in Equation 4 correspond to the saliencies,  $x_i$ , in Equation 2. The pair wise coupling constants,  $c_{ij}$ , (Equation 6) correspond to the affinities,  $a_{ij}$ . By engineering the distances between the magnets if we can modify the coupling constants to match the respective affinities, then the minimum state of the arrangement will give us an approximate solution to the original problem. Figure 1 shows some examples of this placement for two straight lines. For the general case of  $N$  straight lines we rely on the body of work in statistics called Multidimensional Scaling (MDS) [26].

The objective is to find a configuration of points, representing the low-level features, in a 2D space such that the distance between  $i$ -th and  $j$ -th points,  $d_{ij}$ , will be proportional to affinity between the corresponding lines,  $a_{ij}$ . If magnetic cells are placed at these points coordinates then the pairwise interaction between them will be proportional to the given energies, i.e.  $c_{ij} \propto a_{ij}$ . For this process, we look into the rich areas of graph embedding onto planes [27], [28] and multidimensional scaling [26]. The affinity matrix can be considered to represent the adjacency matrix of a weighted graph. The problem then is to embedded the nodes of the graph in the plane in such a way as to preserve an edges weight as Euclidean distance between them. If we allow for distortions of weights of the graphs, this is indeed possible [28], [29]. We have developed an approach based on multidimensional scaling.

Let the matrix  $\Lambda$  be constructed out of given affinities such that:  $\Lambda_{rs} = \frac{1}{(A_{ij})^2}$ . We desire to find the coordinate of each point in a 2D space, which we denote by the matrix of coordinate vector,  $\mathbf{X}_{MDS} = [\mathbf{x}_1, \dots, \mathbf{x}_N]$ , such that

$$(\mathbf{x}_i - \mathbf{x}_j)^T (\mathbf{x}_i - \mathbf{x}_j) = c\Lambda_{rs} \quad (7)$$

$$\mathbf{X}_{MDS}^T \mathbf{X}_{MDS} = -c\frac{1}{2}\mathbf{H}\Lambda\mathbf{H}, \quad \text{where } \mathbf{H} = (\mathbf{I} - \frac{1}{N}\vec{1}\vec{1}^T) \quad (8)$$

with  $\mathbf{I}$  as the identity matrix and  $\vec{1}$  as the vector of ones. This operator  $\mathbf{H}$  is referred to as the centering operator. These coordinates  $\mathbf{X}$  can be arrived at by classical MDS scheme [26]. The solution is based on the singular value

decomposition of the centered distance matrix  $\frac{1}{2}\mathbf{H}\Lambda\mathbf{H} = \mathbf{V}_{MDS}\Delta_{MDS}\mathbf{V}_{MDS}^T$  where  $\mathbf{V}_{MDS}$ ,  $\Delta_{MDS}$  are the eigenvectors and eigenvalues respectively. Assuming that centered distance matrix represents the inner product distances of a Euclidean distance matrix, the coordinates are given by

$$\mathbf{X}_{MDS} = (\mathbf{V}_{MDS}\Delta_{MDS}^{\frac{1}{2}})^T \quad (9)$$

Note that we have dropped the constant of proportionality,  $c$ , since the energy minimizing solutions are invariant to scaling of the original function. Our nano-magnet selection solution is given by the first two rows of  $\mathbf{X}_{MDS}$ ; each column of this matrix gives us the coordinates of the corresponding nano-magnet to consider.

The computational overhead of this synthesis step is linear in the number of the image features. This replaces the complexity of the software solution to the minimization problem.

#### V. FABRICATION PROCESS

A Si wafer was coated with PMMA via a Laurell Technologies WS-400A-8NPP/Lite Spin Processor. A single thin layer of 950 molecular weight PMMA in anisole was spun with a resulting thickness of approximately 120nm. Afterwards it was baked in an oven which provided even heat over the entire wafer to evaporate any residual solvent. The MFC systems were designed using DesignCAD2000 NT. The most effective line spacing, exposure doses, points, and focus were determined by using diagnostic wheel pattern. The sample was then loaded into a JOEL 840m retrofitted with the NPGS lithography system and beam blander for pattern exposure. Subsequently, the sample was unloaded for development in MIBK:Isopropanol 3:1, after which a thin Permalloy film was deposited via a Varian Model 980-2462 Electron Beam Evaporator. A vacuum of about  $2\mu\text{Torr}$  was achieved and evaporation was conducted at a fast rate to reduce contamination. Liftoff was accomplished by placing the sample coated with Permalloy in a heated ultrasonic acetone bath for approximately 15 minutes.

#### VI. READ OUT SCHEME

Figure 2(a) show an example vision problem where we have to find the visually salient subset of straight lines – the three parallel lines in the middle. Figure 2(b) shows the placement of magnets that can be used to solve this problems. The magnetostatic coupling between the magnets match the pairwise affinities of the lines. Each line is represented by a magnet. The white circles represent magnetic nano-disks, enumerated from left to right. The disks were made of permalloy with the thickness much less than the lateral dimensions to force in-plane single domain magnetic dipole moments. The experiment would proceed by applying an external magnetizing field along a particular direction. Afterwards, the field would be removed and the magnetic cells would be allowed to settle into a ground state as shown in Figure 2(c). The natural tendency for magnetic energy is to be minimized, in this case primarily due to magneto-static coupling. It would lead magnets 1-3 to arrange themselves

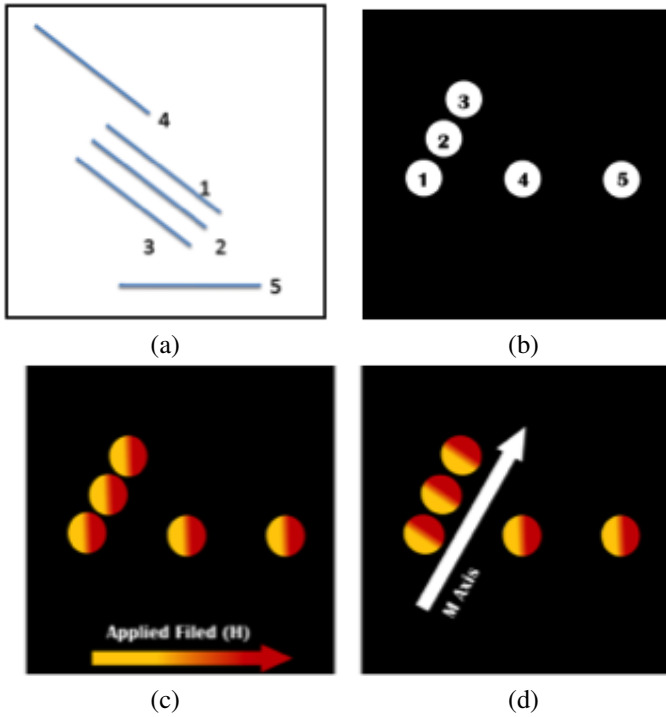


Fig. 2. Read out scheme for validation. (a) The vision problem: find the visually salient lines. (b) Equivalent magnet arrangement with similar Hamiltonian as the vision problem. The disks correspond to the lines. (c) Magnetic initialization of the MFC system into a possible ground state of the system after the external field is removed. (d) If the magnetization is taken along axis M, the magnetization component of nano-disk 1, 2, and 3, will be larger than those of 4 and 5.

in a ferromagnetic fashion. It was also possible for magnet 4 to experience a degree of coupling with magnets 1-3 which could be demonstrated via an anti-ferromagnetic coupling. The coupling exhibited among magnets 1-3 is expected since the inter-spacing distant was smaller than that of magnets 4 and 5. Our experimental measurements do indeed demonstrate this to be the case.

If the magnetization vector of each magnet is taken along axis M as shown in Figure 2(d), the magnetization of magnets 1-3 would be greater than those of magnets 4 and 5. As mentioned previously, this magnetic interaction can be modeled via a quadratic term and is computational intensive but occurs naturally in the physical world, at least on the order of nano-seconds. The magneto-static quadratic term was inversely proportional to the quadratic edge affinity, which determines the salient features of the image. So by setting appropriate affinity thresholds, the edges corresponding to magnets 1-3 would be regarded as a significant grouping, while magnets 4 and 5 would not.

## VII. RESULTS

Figures 3 (a) and (b) show AFM and MFM images of the fabricated cells scanned along three directions. To ensure a single domain magnetic dipole moment and to mitigate vortex states, engineering of the shape anisotropy should be such that the out of plane magnetization component of the vortex core should produce higher demagnetization energy

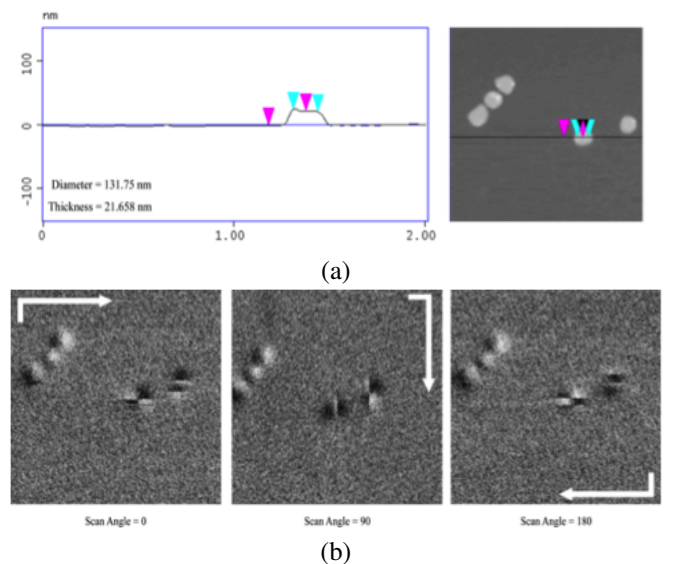


Fig. 3. Fabricated nano-magnets for the vision problem (a) AFM height amplitude image and the fabricated layout of the MFC system. As can be seen in the graph the diameter of the measured nano-disk is 131.75 nm and the thickness is 21.658 nm. (b) Three images show MFM scans of the fabricated arrangement, scanned along different directions.

than that of an in-plane single domain state. The average dimensions of the nano-magnetic disks were approximately 130 nm in diameter and 20 nm thick and are referred to as magnets from left to right. As desired, the three leftmost magnets (1, 2, 3) in all the three different scan directions were in a single domain state, with their magnetic configurations unaltered. This is due to the ferromagnetic dipole-dipole coupling between close neighbors that reduce the susceptibility of the nano-magnets to the stray fields from emanating from the scanning probe.

When analyzing the two right most magnets (magnets 4 and 5), which should be in a decoupled single domain state due to the distance in between their nearest neighbors, we see a pinwheel type magnetization. This was due to sample-probe interaction that caused the nano-disk to flip its magnetization during data acquisition, signifying weak interaction with neighboring elements. Even though the probe tip altered the magnetic state of the nano-disk, the single domain moment could still be extrapolated via the presence of the strong dipoles. By retracing the scanning process of the tip with a scan angle of 0, as shown in Figure 3 (b) by the arrow, where the slow scan is progresses from bottom to top and the fast scan is from left to right, the single domain moment can be followed as it flips throughout the scan. The stray field from the scanning tip was sufficiently strong to flip the magnetization of magnet 4 at least 3 times during the scan because it was decoupled from neighboring elements. A similar process altered the magnetization of magnet 5 and the different flipping behavior is explained by a preferred magnetization axis in the vertical direction. This becomes evident once the scan angle is 90 and the slow scan is in the vertical direction. Magnet 5 flipped once during the scan in

a similar fashion to magnetic 4 with a scan angle of 0.

## VIII. CONCLUSIONS

Magnetic Field-based Computing (MFC) has the potential for offering a unique solution for the quadratic minimization problem. This is unlike other proposals for the use of nano-magnets for Boolean computing. A proof of concept consisting of 5 nano-magnetic disks that related to low-level edge segments of an image has been provided. The proof of concept was fabricated and strictly adhered to a placement algorithm that correlated magneto-static interactions to the edge affinity energies of an image.

## IX. ACKNOWLEDGMENTS

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